Fermionic current densities induced by magnetic flux in a conical space with a circular boundary

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Abstract

We investigate the vacuum expectation value of the fermionic current induced by a magnetic flux in a (2+1)-dimensional conical spacetime in the presence of a circular boundary. On the boundary the fermionic field obeys MIT bag boundary condition. For irregular modes, a special case of boundary conditions at the cone apex is considered, when the MIT bag boundary condition is imposed at a finite radius, which is then taken to zero. We observe that the vacuum expectation values for both charge density and azimuthal current are periodic functions of the magnetic flux with the period equal to the flux quantum whereas the expectation value of the radial component vanishes. For both exterior and interior regions, the expectation values of the current are decomposed into boundary-free and boundary-induced parts. For a massless field the boundary-free part in the vacuum expectation value of the charge density vanishes, whereas the presence of the boundary induces nonzero charge density. Two integral representations are given for the boundary-free part in the case of a massive fermionic field for arbitrary values of the opening angle of the cone and magnetic flux. The behavior of the induced fermionic current is investigated in various asymptotic regions of the parameters. At distances from the boundary larger than the Compton wavelength of the fermion particle, the vacuum expectation values decay exponentially with the decay rate depending on the opening angle of the cone. We make a comparison with the results already known from the literature for some particular cases.

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1 Introduction

Topological defects are inevitably produced during symmetry breaking phase transitions and play an important role in many fields of physics. They appear in different condensed matter systems including superfluids, superconductors and liquid crystals. Moreover, symmetry breaking phase transitions have several cosmological consequences and, within the framework of grand unified

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theories, various types of topological defects are predicted to be formed in the early universe [1]. They provide an important link between particle physics and cosmology. Among various types of topological defects, the cosmic strings are of special interest. They are candidates to produce a number of interesting physical effects, such as the generation of gravitational waves, gamma ray bursts and high-energy cosmic rays. Recently, cosmic strings attract a renewed interest partly because a variant of their formation mechanism is proposed in the framework of brane inflation [2].

In the simplest theoretical model describing the infinite straight cosmic string the spacetime is locally flat except on the string where it has a Dirac-delta shaped Riemann curvature tensor. From the point of view of quantum field theory, the corresponding non-trivial topology induces non-zero vacuum expectation values for several physical observables. Explicit calculations for the geometry of a single idealized cosmic string have been developed for different fields [3]-[24]. Moreover, vacuum polarization effects by higher-dimensional composite topological defects constituted by a cosmic string and global monopole are investigated in Refs. [25] for scalar and fermionic fields. The geometry of a cosmic string in background of de Sitter spacetime has been recently considered in [26].

Another type of vacuum polarization arises in the presence of boundaries. The imposed boundary conditions on quantum fields alter the zero-point fluctuations spectrum and result in additional shifts in the vacuum expectation values of physical quantities. This is the well-known Casimir effect (for a review see [27]). Note that the Casimir forces between material boundaries are presently attracting much experimental attention [28]. In Refs. [29]-[32], both types of sources for the polarization of the vacuum were studied in the cases of scalar, electromagnetic and fermionic fields, namely, a cylindrical boundary and a cosmic string, assuming that the boundary is coaxial with the string. The case of a scalar field was considered in an arbitrary number of spacetime dimensions, whereas the problems for the electromagnetic and fermionic fields were studied in four dimensional spacetime. Continuing in this line of investigation, in the present paper we study the fermionic current induced by a magnetic flux in a (2+1)-dimensional conical space with a circular boundary.

As it is well known, field theoretical models in 2+1 dimensions exhibit a number of interesting effects, such as parity violation, flavour symmetry breaking, fractionalization of quantum numbers (see Refs. [33]-[39]). An important aspect is the possibility of giving a topological mass to the gauge bosons without breaking gauge invariance. Field theories in 2+1 dimensions provide simple models in particle physics and related theories also rise in the long-wavelength description of certain planar condensed matter systems, including models of high-temperature superconductivity. An interesting application of Dirac theory in 2+1 dimensions recently appeared in nanophysics. In a sheet of hexagons from the graphite structure, known as graphene, the long-wavelength description of the electronic states can be formulated in terms of the Dirac-like theory of massless spinors in (2+1)-dimensional spacetime with the Fermi velocity playing the role of speed of light (for a review see Ref. [40]). One-loop quantum effects induced by non-trivial topology of graphene made cylindrical and toroidal nanotubes have been recently considered in Refs. [41]. The vacuum polarization in graphene with a topological defect is investigated in Ref. [42] within the framework of long-wavelength continuum model.

The interaction of a magnetic flux tube with a fermionic field gives rise to a number of interesting phenomena, such as the Aharonov-Bohm effect, parity anomalies, formation of a condensate and generation of exotic quantum numbers. For background Minkowski spacetime, the combined effects of the magnetic flux and boundaries on the vacuum energy have been studied in Refs. [43, 44]. In the present paper we investigate the vacuum expectation value of the fermionic current induced by vortex configuration of a gauge field in a (2+1)-dimensional conical space with a circular boundary. We assume that on the boundary the fermionic field

obeys MIT bag boundary condition. The induced fermionic current is among the most important quantities that characterize the properties of the quantum vacuum. Although the corresponding operator is local, due to the global nature of the vacuum, this quantity carries an important information about the global properties of the background spacetime. In addition to describing the physical structure of the quantum field at a given point, the current acts as the source in the Maxwell equations. It therefore plays an important role in modelling a self-consistent dynamics involving the electromagnetic field.

From the point of view of the physics in the region outside the conical defect core, the geometry considered in the present paper can be viewed as a simplified model for the non-trivial core. This model presents a framework in which the influence of the finite core effects on physical processes in the vicinity of the conical defect can be investigated. In particular, it enables to specify conditions under which the idealized model with the core of zero thickness can be used. The corresponding results may shed light upon features of finite core effects in more realistic models, including those used for defects in crystals and superfluid helium. In addition, the problem considered here is of interest as an example with combined topological and boundary induced quantum effects, in which the vacuum characteristics can be found in closed analytic form.

The organization of the paper is as follows. In the next section we consider the complete set of solutions to the Dirac equation in the region outside a circular boundary on which the field obeys MIT bag boundary condition. Shrinking the radius of the circle to zero we clarify the structure of the eigenspinors for the boundary-free geometry. These eigenspinors are used in Sect. 3 for the evaluation of the vacuum expectation value of the fermionic current density. Two integral representations are provided for the charge density and azimuthal component. In Sect. 4, we consider the vacuum expectation values in the region outside a circular boundary. They are decomposed into boundary-free and boundary-induced parts. Rapidly convergent integral representations for the latter are obtained. Similar investigation for the region inside a circular boundary is presented in Sect. 5. The main results are summarized in Sect. 6. In Appendix A we derive two integral representations for the series involving the modified Bessel functions. These representations are used to obtain the fermionic current densities in the boundary-free geometry. In Appendix B, we compare the results of the present paper, in the special case of a magnetic flux in (2+1)-dimensional Minkowski spacetime, with those from the literature. In Appendix C we show that the special mode does not contribute to the vacuum expectation value of the fermionic current in the region inside a circular boundary.

2 Model and the eigenspinors in the exterior region

In this paper we consider a two-component spinor field ψ, propagating on a $(2+1)$-dimensional background spacetime with a conical singularity described by the line-element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - dr^2 - r^2 d\phi^2, \tag{2.1}$$

where $r \geqslant 0$, $0 \leqslant \phi \leqslant \phi_0$, and the points (r, ϕ) and $(r, \phi + \phi_0)$ are to be identified. We are interested in the change of the vacuum expectation value (VEV) of the fermionic current induced by a magnetic flux in the presence of a circular boundary concentric with the apex of the cone.

The dynamics of a massive spinor field is governed by the Dirac equation

$$i\gamma^\mu(\nabla_\mu + ieA_\mu)\psi - m\psi = 0\ ,\ \nabla_\mu = \partial_\mu + \Gamma_\mu, \tag{2.2}$$

where A_μ is the vector potential for the external electromagnetic field. In Eq. (2.2), $\gamma^\mu = e^\mu_{(a)}\gamma^{(a)}$ are the 2×2 Dirac matrices in polar coordinates and Γ_μ is the spin connection. The latter is

defined in terms of the flat space Dirac matrices, $\gamma^{(a)}$, by the relation

$$\Gamma_\mu = \frac{1}{4}\gamma^{(a)}\gamma^{(b)}e^{\nu}_{(a)}e_{(b)\nu;\mu} \ , \tag{2.3}$$

where ; means the standard covariant derivative for vector fields. In the equations above, $e^{\mu}_{(a)}$, $a = 0, 1, 2$, is the basis tetrad satisfying the relation $e^{\mu}_{(a)}e^{\nu}_{(b)}\eta^{ab} = g^{\mu\nu}$, with η^{ab} being the Minkowski spacetime metric tensor. We assume that the field obeys the MIT bag boundary condition on the circle with radius a:

$$\left(1 + in_\mu\gamma^\mu\right)\psi\big|_{r=a} = 0 \ , \tag{2.4}$$

where n_μ is the outward oriented normal (with respect to the region under consideration) to the boundary. In particular, from Eq. (2.4) it follows that the normal component of the fermion current vanishes at the boundary, $n_\mu\bar{\psi}\gamma^\mu\psi = 0$, with $\bar{\psi} = \psi^\dagger\gamma^0$ being the Dirac adjoint and the dagger denotes Hermitian conjugation. In this section we consider the region $r > a$ for which $n_\mu = -\delta^1_\mu$.

In (2+1)-dimensional spacetime there are two inequivalent irreducible representations of the Clifford algebra. In the first one we may choose the flat space Dirac matrices in the form

$$\gamma^{(0)} = \sigma_3, \ \gamma^{(1)} = i\sigma_1, \ \gamma^{(2)} = i\sigma_2, \tag{2.5}$$

with σ_l being Pauli matrices. In the second representation the gamma matrices can be taken as $\gamma^{(0)} = -\sigma_3$, $\gamma^{(1)} = -i\sigma_1$, $\gamma^{(2)} = -i\sigma_2$. In what follows we use the representation (2.5). The corresponding results for the second representation are obtained by changing the sign of the mass, $m \to -m$. For the basis tetrads we use the representation below:

$$\begin{aligned} e^{\mu}_{(0)} &= (1, 0, 0) \ , \\ e^{\mu}_{(1)} &= (0, \cos(q\phi), -\sin(q\phi)/r) \ , \\ e^{\mu}_{(2)} &= (0, \sin(q\phi), \cos(q\phi)/r) \ , \end{aligned} \tag{2.6}$$

where the parameter q is related to the opening angle of the cone by the relation

$$q = 2\pi/\phi_0. \tag{2.7}$$

With this choice, for the Dirac matrices in the coordinate system given by the line element (2.1), we have the following representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma^1 = i\begin{pmatrix} 0 & e^{-iq\phi} \\ e^{iq\phi} & 0 \end{pmatrix}, \ \gamma^2 = \frac{1}{r}\begin{pmatrix} 0 & e^{-iq\phi} \\ -e^{iq\phi} & 0 \end{pmatrix}. \tag{2.8}$$

Consequently, the Dirac equation takes the form

$$\left[\gamma^\mu\left(\partial_\mu + ieA_\mu\right) + \frac{1-q}{2r}\gamma^1 + im\right]\psi = 0, \tag{2.9}$$

where the term with γ^1 comes from the spin connection.

In what follows we assume the magnetic field configuration corresponding to a magnetic flux located in the region $r < a$. This will be implemented by considering the vector potential in the exterior region, $r > a$, as follows

$$A_\mu = (0, 0, A), \tag{2.10}$$

In Eq. (2.10), $A_2 = A$ is the covariant component of the vector potential in the coordinates (t, r, ϕ). For the so called physical azimuthal component one has $A_\phi = -A/r$. The quantity A is

related to the magnetic flux Φ by the formula $A = -\Phi/\phi_0$. Though the magnetic field strength, corresponding to (2.10), vanishes outside the flux, the non-trivial topology of the background spacetime leads to Aharonov-Bohm-like effects on physical observables. In particular, as it will be seen below, the VEV of the fermionic current depends on the fractional part of the ratio of Φ by the quantum flux, $2\pi/e$.

Decomposing the spinor into upper and lower components, φ_+ and φ_-, respectively, from Eq. (2.9) we get the following equations

$$(\partial_0 \pm im)\,\varphi_\pm \pm ie^{\mp iq\phi}\Big[\partial_1 + \frac{1-q}{2r} \mp \frac{i}{r}(\partial_2 + ieA)\Big]\varphi_\mp = 0. \tag{2.11}$$

From here we find the second-order differential equation for the separate components:

$$\left(\partial_0^2 - \partial_1^2 - \frac{1}{r}\partial_1 - \frac{1}{r^2}\partial_2^2 \mp 2i\frac{c_\pm}{r^2}\partial_2 + \frac{c_\pm^2}{r^2} + m^2\right)\varphi_\pm = 0, \tag{2.12}$$

with the notations $c_\pm = (q-1)/2 \pm eA$.

For the positive energy solutions, the dependence on the time and angle coordinates is in the form $e^{-iEt+iqn\phi}$ with $E > 0$ and $n = 0, \pm 1, \pm 2, \ldots$. Now, from Eq. (2.12), for the radial function we obtain the Bessel equation with the solution in the region $r > a$:

$$\varphi_+ = Z_{|\lambda_n|}(\gamma r)e^{iqn\phi - iEt}, \tag{2.13}$$

where $\gamma \geqslant 0$,

$$E = \sqrt{\gamma^2 + m^2},\ \lambda_n = q(n + \alpha + 1/2) - 1/2, \tag{2.14}$$

and

$$\alpha = eA/q = -e\Phi/2\pi. \tag{2.15}$$

In Eq. (2.13),

$$Z_{|\lambda_n|}(\gamma r) = c_1 J_{|\lambda_n|}(\gamma r) + c_2 Y_{|\lambda_n|}(\gamma r), \tag{2.16}$$

with $J_\nu(x)$ and $Y_\nu(x)$ being the Bessel and Neumann functions. Note that in Eq. (2.15) the parameter α is the magnetic flux measured in units of the flux quantum $\Phi_0 = 2\pi/e$.

The lower component of the spinor, φ_-, is found from Eq. (2.11) and for the positive energy eigenspinors we get

$$\psi_{\gamma n}^{(+)}(x) = e^{iqn\phi - iEt}\begin{pmatrix} Z_{|\lambda_n|}(\gamma r) \\ \epsilon_{\lambda_n}\frac{\gamma e^{iq\phi}}{E+m}Z_{|\lambda_n|+\epsilon_{\lambda_n}}(\gamma r)\end{pmatrix}, \tag{2.17}$$

where $\epsilon_{\lambda_n} = 1$ for $\lambda_n \geqslant 0$ and $\epsilon_{\lambda_n} = -1$ for $\lambda_n < 0$. From the boundary condition (2.4) with $n_\mu = -\delta_\mu^1$ we find

$$Z_{|\lambda_n|}(\gamma a) + \frac{\epsilon_{\lambda_n}\gamma}{E+m}Z_{|\lambda_n|+\epsilon_{\lambda_n}}(\gamma a) = 0. \tag{2.18}$$

This condition relates the coefficients c_1 and c_2 in the linear combination (2.16):

$$\frac{c_2}{c_1} = -\frac{\bar{J}_{|\lambda_n|}^{(-)}(\gamma a)}{\bar{Y}_{|\lambda_n|}^{(-)}(\gamma a)}. \tag{2.19}$$

Here and in what follows we use the notations (the notation with the upper sign is employed below)

$$\bar{f}^{(\pm)}(z) = zf'(z) + (\pm\sqrt{z^2 + \mu^2} \pm \mu - \lambda_n)f(z),\ \mu = ma, \tag{2.20}$$

for a given function $f(z)$.

Hence, outside a circular boundary the positive energy eigenspinors are presented in the form

$$\psi_{\gamma n}^{(+)}(x) = c_0 e^{iqn\phi - iEt} \begin{pmatrix} g_{|\lambda_n|,|\lambda_n|}(\gamma a, \gamma r) \\ \epsilon_{\lambda_n} \frac{\gamma e^{iq\phi}}{E+m} g_{|\lambda_n|,|\lambda_n|+\epsilon_{\lambda_n}}(\gamma a, \gamma r) \end{pmatrix}, \tag{2.21}$$

where

$$g_{\nu,\rho}(x,y) = \bar{Y}_\nu^{(-)}(x) J_\rho(y) - \bar{J}_\nu^{(-)}(x) Y_\rho(y). \tag{2.22}$$

Using the properties of the Bessel functions it can be seen that

$$g_{\nu,\nu}(x,y) = g_{-\nu,-\nu}(x,y),\ g_{\nu,\nu+1}(x,y) = -g_{-\nu,-\nu-1}(x,y). \tag{2.23}$$

Note that the spinor (2.21) is an eigenfunction of the operator $\widehat{J} = -(i/q)\partial_\phi + \sigma_3/2$, with the eigenvalue $j = n + 1/2$, i.e.,

$$\widehat{J}\psi_{\gamma n}^{(+)}(x) = j\psi_{\gamma n}^{(+)}(x),\ j = n + 1/2. \tag{2.24}$$

The coefficient c_0 in Eq. (2.21) is determined from the orthonormalization condition for the eigenspinors:

$$\int_a^\infty dr \int_0^{\phi_0} d\phi\, r \psi_{\gamma n}^{(+)\dagger}(x) \psi_{\gamma' n'}^{(+)}(x) = \delta(\gamma - \gamma')\delta_{nn'}\ . \tag{2.25}$$

The integral over r is divergent when $\gamma' = \gamma$ and, hence, the main contribution comes from the upper limit of the integration. In this case, we can replace the Bessel and Neumann functions, having in the arguments the radial coordinate r, by the corresponding asymptotic expressions for large values of their argument. In this way, for the normalization coefficient we find,

$$c_0^2 = \frac{2E\gamma}{\phi_0(E+m)} \left[\bar{J}_{|\lambda_n|}^{(-)2}(\gamma a) + \bar{Y}_{|\lambda_n|}^{(-)2}(\gamma a) \right]^{-1}. \tag{2.26}$$

The negative energy eigenspinors are constructed in a similar way and they are given by the expression

$$\psi_{\gamma n}^{(-)}(x) = c_0 e^{-iqn\phi + iEt} \begin{pmatrix} \epsilon_{\lambda_n} \frac{\gamma e^{-iq\phi}}{E+m} g_{|\lambda_n|,|\lambda_n|+\epsilon_{\lambda_n}}(\gamma a, \gamma r) \\ g_{|\lambda_n|,|\lambda_n|}(\gamma a, \gamma r) \end{pmatrix}, \tag{2.27}$$

with the same normalization coefficient defined by Eq. (2.26). Note that the positive and negative energy eigenspinors are related by the charge conjugation which can be written as $\psi_{\gamma n}^{(-)} = \sigma_1 \psi_{\gamma n}^{(+)*}$, where the asterisk means complex conjugate.

We can generalize the eigenspinors given above for a more general situation where the spinor field ψ obeys quasiperiodic boundary condition along the azimuthal direction

$$\psi(t, r, \phi + \phi_0) = e^{2\pi i \chi} \psi(t, r, \phi), \tag{2.28}$$

with a constant parameter χ, $|\chi| \leqslant 1/2$. With this condition, the exponential factor in the expressions for the eigenspinors has the form $e^{\pm iq(n+\chi)\phi \mp iEt}$ for the positive and negative energy modes (upper and lower signs respectively). The corresponding expressions for the eigenfunctions are obtained from those given above with the parameter α defined by

$$\alpha = \chi - e\Phi/2\pi. \tag{2.29}$$

The same replacement generalizes the expressions for the VEVs of the fermionic current, given below, for the case of a field with periodicity condition (2.28). The property, that the VEVs depend on the phase χ and on the magnetic flux in the combination (2.29), can also be seen by the gauge transformation $A_\mu = A'_\mu + \partial_\mu \Lambda(x)$, $\psi(x) = \psi'(x) e^{-ie\Lambda(x)}$, with the function $\Lambda(x) = A_\mu x^\mu$. The new function $\psi'(x)$ satisfies the Dirac equation with $A'_\mu = 0$ and the quasiperiodicity condition similar to (2.28) with the replacement $\chi \to \chi' = \chi - e\Phi/2\pi$.

3 Fermionic current in a boundary-free conical space

Before considering the fermionic current in the region outside a circular boundary, in this section we study the case of a boundary-free conical space with an infinitesimally thin magnetic flux placed at the apex of the cone. The corresponding vector potential is given by Eq. (2.10) for $r > 0$. As it is well known, the theory of von Neumann deficiency indices leads to a one-parameter family of allowed boundary conditions in the background of an Aharonov-Bohm gauge field [45]. In this paper, we consider a special case of boundary conditions at the cone apex, when the MIT bag boundary condition is imposed at a finite radius, which is then taken to zero (note that similar approach, with the Atiyah-Patodi-Singer type nonlocal boundary conditions, has been used in Refs. [44] for a magnetic flux in Minkowski spacetime). The VEVs of the fermionic current for other boundary conditions on the cone apex are evaluated in a way similar to that described below. The contribution of the regular modes is the same for all boundary conditions and the results differ by the parts related to the irregular modes.

3.1 Eigenspinors

In order to clarify the structure of the eigenspinors in a boundary-free conical space, we consider the limit $a \to 0$ for Eqs. (2.21) and (2.27). In this limit, using the asymptotic formulae for the Bessel functions for small values of the arguments, for the modes with $j \neq -\alpha$ we find

$$\begin{aligned} \psi^{(+)}_{(0)\gamma j}(x) &= c_0^{(0)} e^{iqj\phi - iEt} \begin{pmatrix} J_{\beta_j}(\gamma r) e^{-iq\phi/2} \\ \frac{\gamma\epsilon_j e^{iq\phi/2}}{E+m} J_{\beta_j+\epsilon_j}(\gamma r) \end{pmatrix}, \\ \psi^{(-)}_{(0)\gamma j}(x) &= c_0^{(0)} e^{-iqj\phi + iEt} \begin{pmatrix} \frac{\gamma\epsilon_j e^{-iq\phi/2}}{E+m} J_{\beta_j+\epsilon_j}(\gamma r) \\ J_{\beta_j}(\gamma r) e^{iq\phi/2} \end{pmatrix}, \end{aligned} \tag{3.1}$$

where

$$\beta_j = q|j+\alpha| - \epsilon_j/2. \tag{3.2}$$

Note that one has $\epsilon_j \beta_j = \lambda_n$. In Eqs. (3.1) and (3.2), we have defined

$$\epsilon_j = \begin{cases} 1, & j > -\alpha \\ -1, & j < -\alpha \end{cases}, \tag{3.3}$$

and the normalization coefficient is given by the expression

$$c_0^{(0)2} = \frac{\gamma}{\phi_0} \frac{E+m}{2E}. \tag{3.4}$$

In the case when $\alpha = N + 1/2$, with N being an integer number, the eigenspinors with $j \neq -\alpha$ are still given by Eqs. (3.1). The eigenspinors for the mode with $j = -\alpha$, obtained from Eqs. (2.21) and (2.27) in the limit $a \to 0$, have the form (3.1) with the replacements

$$\begin{aligned} J_{\beta_j}(z) &\to (E+m) J_{1/2}(z) - \gamma Y_{1/2}(z), \\ J_{\beta_j+\epsilon_j}(z) &\to (E+m) J_{-1/2}(z) - \gamma Y_{-1/2}(z), \end{aligned} \tag{3.5}$$

and $\epsilon_j = -1$. The corresponding normalization coefficient is defined as $c_0^{(0)} = (2E)^{-1}\sqrt{\gamma/\phi_0}$. Taking into account the expressions for the cylinder functions with the orders $\pm 1/2$, the negative energy eigenspinors in this case are written as

$$\psi^{(-)}_{(0)\gamma,-\alpha}(x) = \left(\frac{E+m}{\pi\phi_0 r E}\right)^{1/2} e^{iq\alpha\phi + iEt} \begin{pmatrix} \frac{\gamma e^{-iq\phi/2}}{E+m} \sin(\gamma r - \gamma_0) \\ e^{iq\phi/2} \cos(\gamma r - \gamma_0) \end{pmatrix}, \tag{3.6}$$

where $\gamma_0 = \arccos[\sqrt{(E-m)/2E}]$. Note that the eigenspinors obtained from (3.1) in the limits $\alpha \to (N+1/2)^{\pm}$ do not coincide with (3.6). For the limit from below (above), $\alpha \to (N+1/2)^{-}$ ($\alpha \to (N+1/2)^{+}$), the eigenspinors are given by Eq. (3.6) with the replacement $\gamma_0 \to \pi/2$ ($\gamma_0 \to 0$). Hence, for the bag boundary condition on the cone apex, the eigenspinors in the boundary-free geometry are discontinuous at points $\alpha = N+1/2$. Notice that, in the presence of the circular boundary, the eigenspinors in the region outside the boundary, given by Eqs. (2.21) and (2.27), are continuous.

In general, the fermionic modes in background of the magnetic vortex are divided into two classes, regular and irregular (square integrable) ones. In the problem under consideration, for given q and α, the irregular mode corresponds to the value of j for which $q|j+\alpha| < 1/2$. If we present the parameter α in the form

$$\alpha = \alpha_0 + n_0,\ |\alpha_0| < 1/2, \tag{3.7}$$

being n_0 an integer number, then the irregular mode is present if $|\alpha_0| > (1-1/q)/2$. This mode corresponds to $j = -n_0 - \mathrm{sgn}(\alpha_0)/2$. Note that, in a conical space, under the condition

$$|\alpha_0| \leqslant (1-1/q)/2, \tag{3.8}$$

there are no square integrable irregular modes. As we have already mentioned, there is a one-parameter family of allowed boundary conditions for irregular modes, parametrized with the angle θ, $0 \leqslant \theta < 2\pi$ (see Ref. [45]). For $|\alpha_0| < 1/2$, the boundary condition, used in deriving eigenspinors (3.1), corresponds to $\theta = 3\pi/2$. If α is a half-integer, the irregular mode corresponds to $j = -\alpha$ and for the corresponding boundary condition one has $\theta = 0$. Note that in both cases there are no bound states.

3.2 Vacuum expectation value of the fermionic current

The VEV of the fermionic current, $j^{\mu}(x) = e\bar{\psi}\gamma^{\mu}\psi$, can be evaluated by using the mode sum formula

$$\langle j^{\nu}(x)\rangle = e\sum_j \int_0^{\infty} d\gamma\, \bar{\psi}^{(-)}_{\gamma j}(x)\gamma^{\nu}\psi^{(-)}_{\gamma j}(x), \tag{3.9}$$

where $\sum_j$ means the summation over $j = \pm 1/2, \pm 3/2, \ldots$. In this section we consider this VEV for a conical space in the absence of boundaries. The corresponding quantities will be denoted by subscript 0. For the geometry under consideration, the eigenspinors are given by expressions (3.1). Substituting them into Eq. (3.9) one finds

$$\begin{aligned}
\langle j^0(x)\rangle_0 &= \frac{eq}{4\pi}\sum_j \int_0^{\infty} d\gamma \frac{\gamma}{E}\left[(E-m)J^2_{\beta_j+\epsilon_j}(\gamma r) + (E+m)J^2_{\beta_j}(\gamma r)\right], \\
\langle j^2(x)\rangle_0 &= \frac{eq}{2\pi r}\sum_j \epsilon_j \int_0^{\infty} d\gamma \frac{\gamma^2}{E} J_{\beta_j}(\gamma r)J_{\beta_j+\epsilon_j}(\gamma r),
\end{aligned} \tag{3.10}$$

and the VEV of the radial component vanishes, $\langle j^1(x)\rangle_0 = 0$. In deriving Eqs. (3.10), we have assumed that the parameter α is not a half-integer. When α is equal to a half-integer, the contribution of the mode with $j = -\alpha$ should be evaluated by using eigenspinors (3.6). The contribution for all other j is still given by Eqs. (3.10). As it will be shown below, both these contributions are separately zero and for half-integer values of α the renormalized VEV of the fermionic current vanishes.

In order to regularize expressions (3.10) we introduce a cutoff function $e^{-s\gamma^2}$ with the cutoff parameter $s > 0$. At the end of calculations the limit $s \to 0$ is taken. First let us consider the charge density. The corresponding regularized expectation value is presented in the form

$$\begin{aligned} \langle j^0(x)\rangle_{0,\mathrm{reg}} &= \frac{eqm}{4\pi}\sum_j \int_0^\infty d\gamma \, \frac{\gamma e^{-s\gamma^2}}{\sqrt{\gamma^2+m^2}} \left[J^2_{\beta_j}(\gamma r) - J^2_{\beta_j+\epsilon_j}(\gamma r)\right] \\ &+ \frac{eq}{4\pi}\sum_j \int_0^\infty d\gamma \, \gamma e^{-s\gamma^2} \left[J^2_{\beta_j}(\gamma r) + J^2_{\beta_j+\epsilon_j}(\gamma r)\right]. \end{aligned} \tag{3.11}$$

Using the representation

$$\frac{1}{\sqrt{\gamma^2+m^2}} = \frac{2}{\sqrt{\pi}}\int_0^\infty dt e^{-(\gamma^2+m^2)t^2}, \tag{3.12}$$

we change the order of integrations in the first term of the right-hand side in Eq. (3.11) and use the formula [46]

$$\int_0^\infty d\gamma \, \gamma e^{-s\gamma^2} J^2_\beta(\gamma r) = \frac{1}{2s} e^{-r^2/2s} I_\beta(r^2/2s), \tag{3.13}$$

with $I_\beta(z)$ being the modified Bessel function. The second term on the right of Eq. (3.11) is directly evaluated using Eq. (3.13). As a result, we get the following integral representation for the regularized charge density:

$$\begin{aligned} \langle j^0(x)\rangle_{0,\mathrm{reg}} &= \frac{eqme^{m^2 s}}{2(2\pi)^{3/2}}\sum_j \int_0^{r^2/2s} dz \frac{z^{-1/2}e^{-m^2r^2/2z}}{\sqrt{r^2-2zs}} e^{-z}\left[I_{\beta_j}(z) - I_{\beta_j+\epsilon_j}(z)\right] \\ &+ \frac{eqe^{-r^2/2s}}{8\pi s}\sum_j \left[I_{\beta_j}(r^2/2s) + I_{\beta_j+\epsilon_j}(r^2/2s)\right]. \end{aligned} \tag{3.14}$$

The renormalization procedure for this expression is described below.

Now we turn to the azimuthal component of the fermionic current. Using the relation

$$zJ_{\beta_j+\epsilon_j}(z) = \beta_j J_{\beta_j}(z) - \epsilon_j z J'_{\beta_j}(z), \tag{3.15}$$

we write the corresponding regularized expression in the form

$$\langle j^2(x)\rangle_{0,\mathrm{reg}} = \frac{eq}{2\pi r^2}\sum_j (\epsilon_j\beta_j - r\partial_r/2)\int_0^\infty d\gamma \, \gamma \frac{e^{-s\gamma^2}J^2_{\beta_j}(xr)}{\sqrt{\gamma^2+m^2}}. \tag{3.16}$$

In a way similar to that we have used for the first term in the right-hand side of Eq. (3.11), the azimuthal current is presented in the form

$$\langle j^2(x)\rangle_{0,\mathrm{reg}} = \frac{eqe^{m^2 s}}{(2\pi)^{3/2}r^2}\sum_j \int_0^{r^2/2s} dz \frac{z^{1/2}e^{-m^2r^2/2z}}{\sqrt{r^2-2zs}} e^{-z}\left[I_{\beta_j}(z) - I_{\beta_j+\epsilon_j}(z)\right]. \tag{3.17}$$

In deriving this representation we employed the relation

$$(\epsilon_j\beta_j - r\partial_r/2)\, e^{-r^2 y} I_{\beta_j}(r^2 y) = ze^{-z}\left[I_{\beta_j}(z) - I_{\beta_j+\epsilon_j}(z)\right]_{z=r^2 y}, \tag{3.18}$$

for the modified Bessel function.

The expressions of the regularized VEVs for both charge density and azimuthal current are expressed in terms of the series

$$\mathcal{I}(q,\alpha,z) = \sum_j I_{\beta_j}(z). \tag{3.19}$$

If we present the parameter α related to the magnetic flux as (3.7), then Eq. (3.19) is written in the form

$$\mathcal{I}(q,\alpha,z)=\sum_{n=0}^{\infty}\left[I_{q(n+\alpha_0+1/2)-1/2}(z)+I_{q(n-\alpha_0+1/2)+1/2}(z)\right], \tag{3.20}$$

which explicitly shows the independence of the series on n_0. Note that for the second series appearing in the expressions for the VEVs of the fermionic current we have

$$\sum_j I_{\beta_j+\epsilon_j}(z)=\mathcal{I}(q,-\alpha_0,z). \tag{3.21}$$

We conclude that the VEVs of the fermionic current depend on α_0 alone and, hence, these VEVs are periodic functions of α with period 1.

When the parameter α is equal to a half-integer, that means $|\alpha_0|=1/2$, the contribution to the VEVs from the modes with $j\neq-\alpha$ is still given by Eqs. (3.14) and (3.17). It is easily seen that for the case under consideration $\sum_{j\neq-\alpha}\left[I_{\beta_j}(x)-I_{\beta_j+\epsilon_j}(x)\right]=0$. The contribution of the mode $j=-\alpha$ is evaluated by using eigenspinors (3.6). A simple evaluation shows that this contribution vanishes as well. As regards the second term on the right-hand side of Eq. (3.14), below it will be shown that this term does not contribute to the renormalized VEV of the charge density. Hence, the renormalized VEVs for both charge density and azimuthal current vanish in the case when the parameter α is equal to a half-integer. Note that in the limit $\alpha_0\to\pm1/2$, $|\alpha_0|<1/2$, one has

$$\lim_{\alpha_0\to\pm1/2}\sum_{\delta=\pm1}\delta\mathcal{I}(q,\delta\alpha_0,z)=\mp\sqrt{2/\pi z}e^{-z}, \tag{3.22}$$

and the expressions for the regularized VEVs are discontinuous at $\alpha_0=\pm1/2$.

3.3 Renormalized VEV in a special case

Before further considering the fermionic current for the general case of the parameters characterizing the conical structure and the magnetic flux, we study a special case, which allows us to obtain simple expressions. It has been shown in [8, 9, 12] that when the parameter q is an integer number, the scalar Green function in four-dimensional cosmic string spacetime can be expressed as a sum of q images of the Minkowski spacetime function. Also, recently the image method was used in [22] to provide closed expressions for the massive scalar Green functions in a higher-dimensional cosmic string spacetime. The mathematical reason for the use of the image method in these applications is because the order of the modified Bessel functions that appear in the expressions for the VEVs becomes an integer number. As we have seen, for the fermionic case the order of the Bessel function depends, besides on the integer angular quantum number $n=j-1/2$, also on the factor $(q-1)/(2q)$ which comes from the spin connection. However, considering a charged fermionic field in the presence of a magnetic flux, an additional term will be present, the factor α. In the special case with q being an integer and

$$\alpha=1/2q-1/2, \tag{3.23}$$

the orders of the modified Bessel functions in Eqs. (3.14) and (3.17) become integer numbers: $\beta_j=q|n|$, $j=n+1/2$. In this case the series over n is summarized explicitly by using the formula [46]

$$\sideset{}{'}\sum_{n=0}^{\infty}I_{qn}(x)=\frac{1}{2q}\sum_{k=0}^{q-1}e^{x\cos(2\pi k/q)}, \tag{3.24}$$

where the prime on the summation sign means that the term $n=0$ should be halved.

By making use of Eq. (3.24), for the regularized VEV of charge density we find

$$
\begin{aligned}
\langle j^0(x)\rangle_{0,\mathrm{reg}} &= \frac{eme^{m^2 s}}{(2\pi)^{3/2}}\sum_{k=1}^{q-1}\sin^2(\pi k/q)\int_0^{r^2/2s}dz\frac{z^{-1/2}e^{-m^2r^2/2z}}{\sqrt{r^2-2zs}}e^{-2z\sin^2(\pi k/q)} \\
&+\frac{e}{4\pi s}\sum_{k=0}^{q-1}\cos^2(\pi k/q)e^{-2(r^2/2s)\sin^2(\pi k/q)}. \qquad (3.25)
\end{aligned}
$$

In the limit $s \to 0$, the only divergent contribution to the right-hand side comes from the term $k = 0$. This term does not depend on the parameter q and is the same as in the Minkowski spacetime in the absence of the magnetic flux. Subtracting the $k = 0$ term, then we take the limit $s \to 0$. The integral is expressed in terms of the Macdonald function $K_{1/2}(2mr\sin(\pi k/q))$ and for the renormalized charge density we find:

$$
\langle j^0(x)\rangle_{0,\mathrm{ren}} = \frac{em}{4\pi r}\sum_{k=1}^{q-1}\sin(\pi k/q)e^{-2mr\sin(\pi k/q)}. \qquad (3.26)
$$

Note that the second term on the right-hand side of Eq. (3.14) does not contribute to the renormalized VEV. Expression (3.26) coincides with the result of Ref. [24] obtained by using the Green function approach.

In a similar way, using (3.24), for the regularized VEV of the azimuthal current we get the formula

$$
\langle j^2(x)\rangle_{0,\mathrm{reg}} = \frac{e}{\pi r^2}\frac{e^{m^2s}}{\sqrt{2\pi}}\sum_{k=1}^{q-1}\sin^2(\pi k/q)\int_0^{r^2/2s}dz\frac{z^{1/2}e^{-m^2r^2/2z}}{\sqrt{r^2-2zs}}e^{-2z\sin^2(\pi k/q)}. \qquad (3.27)
$$

This expression is finite in the limit $s \to 0$ and for the renormalized VEV one finds

$$
\langle j^2(x)\rangle_{0,\mathrm{ren}} = \frac{e}{8\pi r^3}\sum_{k=1}^{q-1}\frac{1+2mr\sin(\pi k/q)}{\sin(\pi k/q)}e^{-2mr\sin(\pi k/q)}. \qquad (3.28)
$$

After the coordinate transformation $\phi' = q\phi$, this expression coincides with the result given in Ref. [24]. Note that for both charge density and azimuthal current one has $\langle j^\nu(x)\rangle_{0,\mathrm{ren}}/e \geqslant 0$. As expected the renormalized current densities decay exponentially at distances larger than the Compton wavelength of the fermionic particle. In Fig. 1 the VEVs of charge density and azimuthal current are plotted versus mr for different values of q. The corresponding values of the parameter α are found from Eq. (3.23).

3.4 General case

Now we turn to the general case for the parameters q and α. As it follows from Eqs. (3.14) and (3.17), we need to evaluate the integrals

$$
\int_0^{r^2/2s}dx\frac{x^{\pm 1/2}e^{-m^2r^2/2x}}{\sqrt{r^2-2xs}}e^{-x}\left[\mathcal{I}(q,\alpha_0,x)-\mathcal{I}(q,-\alpha_0,x)\right], \qquad (3.29)
$$

in the limit $s \to 0$, with the function $\mathcal{I}(q,\alpha_0,x)$ defined by Eq. (3.19). Two alternative integral representations for this series are given in Appendix A. In order to provide an integral representation for the VEV of fermionic current, we first consider the representation (A.6). In the limit $s \to 0$, the only divergent contributions to the integrals in Eq. (3.29) with separate functions

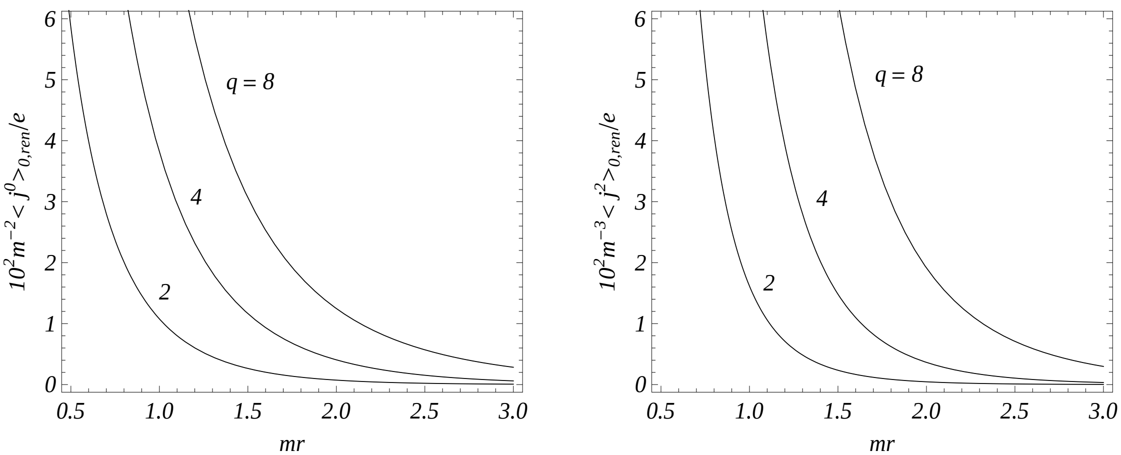

Figure 1: VEVs of the charge density (left plot) and the azimuthal current (right plot) in a boundary-free conical space, as functions of the parameter mr for the special case of integer values of q with the magnetic flux defined by Eq. (3.23).

$\mathcal{I}(q, \pm\alpha_0, x)$ come from the first term in the right-hand side of Eq. (A.6). This term does not depend on α_0 and, consequently, it is cancelled in the evaluation of the integral in Eq. (3.29).

Hence, we see that the regularized expression (3.17) for the VEV of azimuthal current is finite in the limit $s \to 0$. For the corresponding renormalized quantity we find

$$\langle j^2(x)\rangle_{0,\mathrm{ren}} = \frac{eqr^{-3}}{(2\pi)^{3/2}} \int_0^\infty dz\, z^{1/2} e^{-m^2r^2/2z-z} \sum_{\delta=\pm1} \delta\mathcal{I}(q, \delta\alpha_0, z). \tag{3.30}$$

where the function $\mathcal{I}(q, \alpha_0, z)$ is given by the integral representation (A.6). The VEV of the azimuthal current is a periodical function of the magnetic flux with a period equal to magnetic flux quantum Φ_0. It is an odd function of the parameter α_0 defined by Eq. (3.7). From Eq. (A.6) it is seen that the integrand in Eq. (3.30) decays exponentially for large values of z. Substituting the integral representation (A.6) into Eq. (3.30) and changing the order of integrations in the part with the last term of Eq. (A.6), the integral over z is expressed in terms of the Macdonald function $K_{3/2}(y)$.

As a result, for the renormalized VEV of the azimuthal component we find the following expression

$$\begin{aligned}\langle j^2(x)\rangle_{0,\mathrm{ren}} = \frac{e}{4\pi r^3}\Big\{ &\sum_{l=1}^{p} \frac{(-1)^l \sin(2\pi l\alpha_0)}{\sin^2(\pi l/q)} \frac{1+2mr\sin(\pi l/q)}{e^{2mr\sin(\pi l/q)}} \\ &-\frac{q}{4\pi}\int_0^\infty dy \frac{\sum_{\delta=\pm1}\delta f(q,\delta\alpha_0,y)}{\cosh(qy)-\cos(q\pi)} \frac{1+2mr\cosh(y/2)}{\cosh^3(y/2)e^{2mr\cosh(y/2)}}\Big\},\end{aligned} \tag{3.31}$$

where p is an integer defined by $2p < q < 2p+2$ and the function $f(q, \alpha_0, y)$ is given by Eq. (A.5). In the case $q = 2p$, the additional term

$$-(-1)^{q/2}\sin(\pi q\alpha_0)e^{-2mr}(1/2+mr), \tag{3.32}$$

should be added to the expression in the figure braces of Eq. (3.31). For $1 \leqslant q < 2$, only the integral term remains. The difference of the functions $f(q, \alpha_0, y)$, appearing in the integrand,

can also be written in the form

$$\sum_{\delta=\pm 1} \delta f(q, \delta\alpha_0, y) = 2\cosh(y/2) \sum_{\delta=\pm 1} \delta \cos\left[q\pi\left(1/2 - \delta\alpha_0\right)\right] \cosh\left[q\left(1/2 + \delta\alpha_0\right) y\right]. \tag{3.33}$$

Note that for $q = 2p$ the integrand in the last term of Eq. (3.31) is finite at $y = 0$.

In the massless limit the expression in the figure braces of Eq. (3.31) does not depend on the radial coordinate r and the renormalized VEV of the azimuthal current behaves as $1/r^3$. For a massive field, at distances larger than the Compton wavelength of the spinor particle, $mr \gg 1$, the VEV of the azimuthal current is suppressed by the factor e^{-2mr} for $1 \leqslant q < 2$ and by the factor $e^{-2mr\sin(\pi/q)}$ for $q \geqslant 2$. In the limit $mr \ll 1$, the leading term in the corresponding asymptotic expansion coincides with the VEV for a massless field. In Fig. 2 we plot the VEV of the azimuthal current for a massless fermionic field as a function of the magnetic flux for several values of the parameter q (numbers near the curves). In the limit $\alpha_0 \to \pm 1/2$, $|\alpha_0| < 1/2$, the VEV of the azimuthal current is obtained using relation (3.22). From Eq. (3.30) one finds:

$$\lim_{\alpha_0 \to \pm 1/2} \langle j^2(x) \rangle_{0,\mathrm{ren}} = \mp \frac{eqm}{2\pi^2 r^2} K_1(2mr), \tag{3.34}$$

where $K_\nu(z)$ is the Macdonald function.

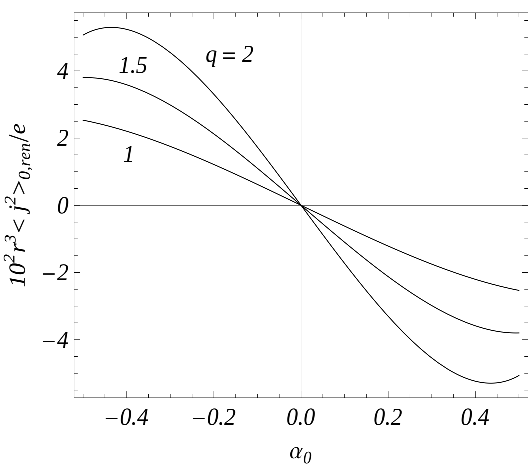

Figure 2: The VEV of the azimuthal current in a boundary-free conical space as a function of the magnetic flux for a massless fermionic field.

Now we turn to the VEV of the charge density. In the corresponding regularized expression (3.14), the first term on the right-hand side with the integral is finite in the limit $s \to 0$. The second term is written in the form

$$\frac{eqe^{-r^2/2s}}{8\pi s} \sum_{\delta=\pm 1} \mathcal{I}(q, \delta\alpha_0, r^2/2s). \tag{3.35}$$

Taking into account Eq. (A.6), we see that the only nonzero contribution to this term comes from the first term in the right-hand side of Eq. (A.6). This contribution diverges and does not depend on the parameters q and α_0. The divergence is the same as in the Minkowski spacetime in the absence of the magnetic flux and is subtracted in the renormalization procedure. As a result, for the renormalized VEV of the charge density we get the formula

$$\langle j^0(x) \rangle_{0,\mathrm{ren}} = \frac{eqm}{2(2\pi)^{3/2} r} \int_0^\infty dz\, z^{-1/2} e^{-m^2 r^2/2z - z} \sum_{\delta=\pm 1} \delta \mathcal{I}(q, \delta\alpha_0, z). \tag{3.36}$$

This VEV is a periodical function of the magnetic flux with a period equal to the magnetic flux quantum. For $2p < q < 2p+2$, with p being an integer, using the integral representation (A.6), the renormalized VEV is presented in the form

$$\langle j^0(x)\rangle_{0,\mathrm{ren}} = \frac{em}{2\pi r}\Big\{\sum_{l=1}^{p}(-1)^l \sin(2\pi l\alpha_0)e^{-2mr\sin(\pi l/q)} - \frac{q}{4\pi}\int_0^\infty dy \frac{\sum_{\delta=\pm1}\delta f(q,\delta\alpha_0,y)}{\cosh(qy)-\cos(q\pi)}\frac{e^{-2mr\cosh(y/2)}}{\cosh(y/2)}\Big\}. \tag{3.37}$$

When $q = 2p$, the additional term

$$-(-1)^{q/2}\sin(\pi q\alpha_0)e^{-2mr}/2, \tag{3.38}$$

should be added to the expression in the figure braces on the right-hand side of Eq. (3.31). As in the case of the azimuthal current, the renormalized VEV (3.36) is an odd function of the parameter α_0. Note that in a boundary-free conical space the charge density for a massless field vanishes at points outside the magnetic flux. For a massive field, at distances larger than the Compton wavelength, $mr \gg 1$, the renormalized charge density is suppressed by the factor e^{-2mr} for $1 \leqslant q < 2$ and by the factor $e^{-2mr\sin(\pi/q)}$ for $q \geqslant 2$.

Though the charge density given by Eq. (3.36) diverges at $r = 0$, this divergence is integrable and the total fermionic charge

$$Q = \phi_0\int_0^\infty dr\, r\langle j^0(x)\rangle_{0,\mathrm{ren}} = \frac{e}{4}\int_0^\infty dx\, e^{-x}\sum_{\delta=\pm1}\delta\mathcal{I}(q,\delta\alpha_0,x), \tag{3.39}$$

is finite. In the form given by Eq. (3.39), the integrals with $\delta = 1$ and $\delta = -1$ diverge separately and we cannot change the order of the integration an summation over δ. In order to overcome this difficulty, we write the integral as $\int_0^\infty dx\, e^{-x}\cdots = \lim_{s\to1^+}\int_0^\infty dx\, e^{-sx}\cdots$. With this representation, evaluating the integrals with separate δ for $s > 1$ and taking the limit $s \to 1$, one finds

$$Q = -e\alpha_0/2. \tag{3.40}$$

This result for a conical space was previously obtained in Ref. [14]. As we see, the total charge does not depend on the angle deficit of the conical space. This property is a consequence of the fact that the total charge is a topologically invariant quantity depending only on the net flux.

In the absence of the angle deficit one has $q = 1$ and the expressions for the VEVs of the fermionic current are simplified to

$$\begin{aligned}\langle j^0(x)\rangle_{0,\mathrm{ren}} &= -\frac{em\sin(\pi\alpha_0)}{2\pi^2 r}\int_0^\infty dz\,\frac{\cosh(2\alpha_0 z)}{\cosh z}e^{-2mr\cosh z},\\ \langle j^2(x)\rangle_{0,\mathrm{ren}} &= -\frac{e\sin(\pi\alpha_0)}{4\pi^2 r^3}\int_0^\infty dz\,\frac{\cosh(2\alpha_0 z)}{\cosh^3 z}\left(1+2mr\cosh z\right)e^{-2mr\cosh z}.\end{aligned} \tag{3.41}$$

Alternative expressions for the VEVs of the charge density and azimuthal current in (2+1)-dimensional Minkowski spacetime in the presence of a magnetic flux were given in Ref. [35] (see also [38] for the general case of a one-parameter family of boundary conditions at the origin). We compare these expressions with the formulae obtained in the present paper in Appendix B.

In Fig. 3, the VEVs of the charge density (left panel) and azimuthal current (right panel) are plotted as functions of the magnetic flux for a massive fermionic field in a conical space with $\phi_0 = \pi$. In the limit $\alpha_0 \to \pm1/2$, $|\alpha_0| < 1/2$, for the azimuthal current one has Eq. (3.34). For the charge density the limiting values are given by

$$\lim_{\alpha_0\to\pm1/2}\langle j^0(x)\rangle_{0,\mathrm{ren}} = \mp\frac{eqm}{2\pi^2 r}K_0(2mr). \tag{3.42}$$

Both charge density and azimuthal current exhibit the jump structure at half-integer values for the ratio of the magneitc flux to the flux quantum (for a similar structure of the persistent currents in carbon nanotube based rings see, for example, Refs. [47]).

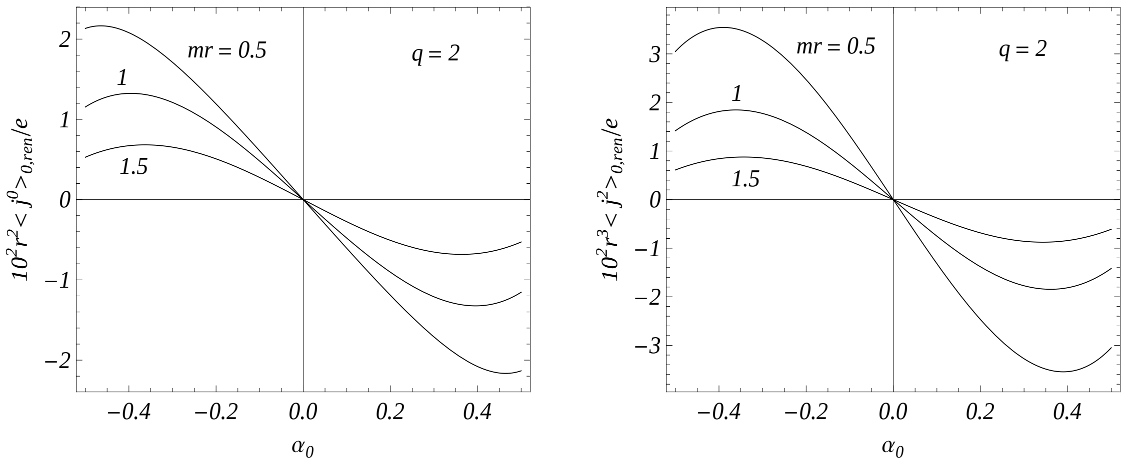

Figure 3: The VEVs of the charge density (left panel) and the azimuthal current (right panel) as functions of the magnetic flux for a massive fermionic field in a boundary-free conical space with $q = 2$.

Alternative expressions for the VEVs of the fermionic current are obtained by using the integral representation (A.15) for the functions $\mathcal{I}(q, \pm\alpha_0, z)$. We start with the charge density. The corresponding regularized expression is given by Eq. (3.14). As we have already noticed, the first term on the right of this formula is finite in the limit $s \to 0$. After the application of (A.15) to the series in the second term, we see that the parts corresponding to the first and last terms in the right-hand side of (A.15) vanish in the limit $s \to 0$ due to the exponential decay of the Macdonald functions. The only term which survives in the limit $s \to 0$ is the part corresponding to the second term in the right-hand side of Eq. (A.15). In the expression for the fermionic current this term is multiplied by q and hence it does not depend on the angle deficit and on the magnetic flux. So, this term is the same as in the case of Minkowski spacetime in the absence of the magnetic flux and is subtracted in the renormalization procedure. As a result, for the renormalized charge density one finds the expression below:

$$\begin{aligned}\langle j^0(x)\rangle_{0,\mathrm{ren}} &= -\frac{2em}{(2\pi)^{5/2}r}\int_0^\infty dz\, z^{-1/2}e^{-m^2r^2/2z-z} \\ &\times\Big[\mathrm{sgn}(\alpha_0)qB(q(|\alpha_0|-1/2)+1/2,z)+2\int_0^\infty dy\, K_{iy}(z)g(q,\alpha_0,y)\Big],\end{aligned} \tag{3.43}$$

where

$$B(y,z) = \begin{cases} 0, & y \leqslant 0, \\ \sin(\pi y)K_y(z) & y > 0, \end{cases} \tag{3.44}$$

and we have defined the function

$$g(q,\alpha_0,y) = \sum_{\delta=\pm1}\mathrm{Re}\left[\frac{\delta\sinh(y\pi)}{e^{2\pi(y+i|q\delta\alpha_0-1/2|)/q}+1}\right]. \tag{3.45}$$

For the VEV of the azimuthal current, in a similar way, we get the representation

$$\langle j^2(x)\rangle_{0,\text{ren}} = -\frac{4er^{-3}}{(2\pi)^{5/2}}\int_0^\infty dz\, z^{1/2}e^{-m^2r^2/2z-z}$$
$$\times\Big[\text{sgn}(\alpha_0)qB(q(|\alpha_0|-1/2)+1/2,z)+2\int_0^\infty dy\,K_{iy}(z)g(q,\alpha_0,y)\Big]. \quad (3.46)$$

Note that under the condition (3.8) there are no square integrable irregular modes and, in this case, the first terms in the square brackets of Eqs. (3.43) and (3.46) vanish.

In the special case $q=1$ we see that $g(1,\alpha_0,y)=0$, and for the VEVs of the fermionic current we obtain the formulae

$$\begin{aligned}\langle j^0(x)\rangle_{0,\text{ren}} &= -\frac{2em\sin(\pi\alpha_0)}{(2\pi)^{5/2}r}\int_0^\infty dz\, e^{-m^2r^2/2z-z}\frac{K_{\alpha_0}(z)}{\sqrt{z}},\\ \langle j^2(x)\rangle_{0,\text{ren}} &= -\frac{4e\sin(\pi\alpha_0)}{(2\pi)^{5/2}r^3}\int_0^\infty dz\,\sqrt{z}e^{-m^2r^2/2z-z}K_{\alpha_0}(z).\end{aligned} \quad (3.47)$$

In the limit $\alpha_0\to\pm 1/2$ we recover results (3.34) and (3.42). In Appendix B, we show the equivalence of these expressions to the ones previously given in the literature for the fermionic densities induced by a magnetic flux in (2+1)-dimensional Minkowski spacetime.

In the discussion above we used the irreducible representation of the Clifford algebra corresponding to Eq. (2.5). For the second representation the renormalized VEV of the azimuthal current is given by the same expressions, whereas the expressions for the VEV of the renormalized charge density change the sign. Consequently, the total induced charge (3.40) changes the sign as well.

4 Induced fermionic current in the exterior region

Now we turn to the investigation of the induced fermionic current in the presence of a circular boundary at $r=a$ with boundary condition (2.4). The corresponding VEV is evaluated using the mode sum formula (3.9) with the eigenspinors given by Eq. (2.27). For the further discussion it is convenient to write these eigenspinors in an equivalent form by using the properties (2.23):

$$\begin{aligned}\psi^{(+)}_{\gamma j}(x) &= c_0e^{iqj\phi-iEt}\begin{pmatrix} g_{\beta_j,\beta_j}(\gamma a,\gamma r)e^{-iq\phi/2}\\ \frac{\gamma\epsilon_j e^{iq\phi/2}}{E+m}g_{\beta_j,\beta_j+\epsilon_j}(\gamma a,\gamma r)\end{pmatrix},\\ \psi^{(-)}_{\gamma j}(x) &= c_0e^{-iqj\phi+iEt}\begin{pmatrix} \frac{\gamma\epsilon_j e^{-iq\phi/2}}{E+m}g_{\beta_j,\beta_j+\epsilon_j}(\gamma a,\gamma r)\\ g_{\beta_j,\beta_j}(\gamma a,\gamma r)e^{iq\phi/2}\end{pmatrix},\end{aligned} \quad (4.1)$$

where c_0 is given by Eq. (2.26) with the replacement $|\lambda_n|\to\beta_j$. In Eq. (4.1), ϵ_j is defined by (3.3) for $j\neq-\alpha$ and $\epsilon_j=-1$ for $j=-\alpha$. We could also obtain the representation (4.1), by taking instead of Eq. (2.16) the linear combination of the functions $J_{\beta_j}(\gamma r)$ and $Y_{\beta_j}(\gamma r)$. Note that the corresponding barred notation in expression (2.22) may also be written in the form

$$\bar{F}^{(\pm)}_{\beta_j}(z) = -\epsilon_j zF_{\beta_j+\epsilon_j}(z)\pm(\sqrt{z^2+\mu^2}+\mu)F_{\beta_j}(z), \quad (4.2)$$

with $F=J,Y$ and $\mu=ma$.

As in the boundary-free case, the VEV of the radial component vanishes and for the charge density and the azimuthal current we have the expressions

$$\begin{aligned}\langle j^0(x)\rangle &= \frac{eq}{4\pi}\sum_j \int_0^\infty d\gamma \frac{\gamma}{E}\frac{(E-m)g^2_{\beta_j,\beta_j+\epsilon_j}(\gamma a,\gamma r)+(E+m)g^2_{\beta_j,\beta_j}(\gamma a,\gamma r)}{\bar{J}^{(-)2}_{\beta_j}(\gamma a)+\bar{Y}^{(-)2}_{\beta_j}(\gamma a)},\\ \langle j^2(x)\rangle &= \frac{eq}{2\pi r}\sum_j \epsilon_j \int_0^\infty d\gamma \frac{\gamma^2}{E}\frac{g_{\beta_j,\beta_j}(\gamma a,\gamma r)g_{\beta_j,\beta_j+\epsilon_j}(\gamma a,\gamma r)}{\bar{J}^{(-)2}_{\beta_j}(\gamma a)+\bar{Y}^{(-)2}_{\beta_j}(\gamma a)}.\end{aligned}\tag{4.3}$$

Here, as before, the summation goes over $j=\pm 1/2,\pm 3/2,\ldots$. Equivalent forms are obtained using the eigenspinors (2.27). From Eq. (4.3) it follows that the fermionic current is a periodic function of α with the period equal to 1. We assume that a cutoff function is introduced without explicitly writing it. The specific form of this function is not important for the discussion below.

In the presence of the boundary, the VEV of the fermionic current can be decomposed as

$$\langle j^\nu(x)\rangle = \langle j^\nu(x)\rangle_0 + \langle j^\nu(x)\rangle_{\rm b},\tag{4.4}$$

where $\langle j^\nu(x)\rangle_{\rm b}$ is the part induced by the boundary. In order to extract the latter explicitly, we subtract from Eq. (4.3) the VEVs when the boundary is absent. If the ratio of the magnetic flux to the flux quantum is not a half-integer, the boundary-free parts are given by Eq. (3.10). In this case, for the evaluation of the difference, in the expression of the charge density we use the identity

$$\frac{g^2_{\beta_j,\lambda}(x,y)}{\bar{J}^{(-)2}_{\beta_j}(x)+\bar{Y}^{(-)2}_{\beta_j}(x)} - J^2_\lambda(y) = -\frac{1}{2}\sum_{l=1,2}\frac{\bar{J}^{(-)}_{\beta_j}(x)}{\bar{H}^{(-,l)}_{\beta_j}(x)}H^{(l)2}_\lambda(y),\tag{4.5}$$

with $\lambda=\beta_j,\beta_j+\epsilon_j$, and with the Hankel functions $H^{(l)}_\nu(x)$. The expression for the boundary-induced part in the VEV of the charge density takes the form

$$\begin{aligned}\langle j^0(x)\rangle_{\rm b} = -\frac{eq}{8\pi}\sum_j\sum_{l=1,2}\int_0^\infty d\gamma\frac{\gamma}{E}\frac{\bar{J}^{(-)}_{\beta_j}(\gamma a)}{\bar{H}^{(-,l)}_{\beta_j}(\gamma a)}\\ \times\left[(E-m)H^{(l)2}_{\beta_j+\epsilon_j}(\gamma r)+(E+m)H^{(l)2}_{\beta_j}(\gamma r)\right].\end{aligned}\tag{4.6}$$

Now, in the complex plane γ, we rotate the integration contour by the angle $\pi/2$ for the term with $l=1$ and by the angle $-\pi/2$ for the term with $l=2$. The integrals over the segments $(0,im)$ and $(0,-im)$ cancel each other and, introducing the modified Bessel functions, we get the following expression

$$\begin{aligned}\langle j^0(x)\rangle_{\rm b} = -\frac{eq}{2\pi^2}\sum_j\int_m^\infty dz\, z\\ \times\Big\{m\frac{K^2_{\beta_j}(zr)+K^2_{\beta_j+\epsilon_j}(zr)}{\sqrt{z^2-m^2}}{\rm Re}\left[I^{(-)}_{\beta_j}(za)/K^{(-)}_{\beta_j}(za)\right]\\ -\left[K^2_{\beta_j}(zr)-K^2_{\beta_j+\epsilon_j}(zr)\right]{\rm Im}\left[I^{(-)}_{\beta_j}(za)/K^{(-)}_{\beta_j}(za)\right]\Big\},\end{aligned}\tag{4.7}$$

where we use the notation (the notation with the upper sign is used in the next section)

$$F^{(\pm)}(z) = zF'(z)+\left(\pm\mu\pm i\sqrt{z^2-\mu^2}-\epsilon_j\beta_j\right)F(z).\tag{4.8}$$

The ratio in the integrand of Eq. (4.7) can also be written in the form

$$\frac{I_{\beta_j}^{(-)}(x)}{K_{\beta_j}^{(-)}(x)}=\frac{W_{\beta_j,\beta_j+\epsilon_j}^{(-)}(x)+i\sqrt{1-\mu^2/x^2}}{x[K_{\beta_j}^2(x)+K_{\beta_j+\epsilon_j}^2(x)]+2\mu K_{\beta_j}(x)K_{\beta_j+\epsilon_j}(x)}, \tag{4.9}$$

with the notation

$$\begin{aligned} W_{\beta_j,\beta_j+\epsilon_j}^{(\pm)}(x) &= x\left[I_{\beta_j}(x)K_{\beta_j}(x)-I_{\beta_j+\epsilon_j}(x)K_{\beta_j+\epsilon_j}(x)\right] \\ &\quad \pm\mu\left[I_{\beta_j+\epsilon_j}(x)K_{\beta_j}(x)-I_{\beta_j}(x)K_{\beta_j+\epsilon_j}(x)\right]. \end{aligned} \tag{4.10}$$

The real and imaginary parts appearing in Eq. (4.7) are easily obtained from Eq. (4.9). Note that under the change $\alpha\to-\alpha$, $j\to-j$, we have $\beta_j\to\beta_j+\epsilon_j$, $\beta_j+\epsilon_j\to\beta_j$. From here it follows that the real/imaginary part in Eq. (4.9) is an odd/even function under this change. Now, from Eq. (4.7) we see that the boundary-induced part in the VEV is an odd function of α. When α is a half-integer, in the term of Eq. (4.7) with $j=-\alpha$ the orders of the modified Bessel functions are equal to $\pm1/2$. Using the expressions of these functions in terms of the elementary functions, we can see that the corresponding integral vanishes. For the terms with $j\neq-\alpha$, the contributions of $j<-\alpha$ and $j>-\alpha$ to the right-hand side of Eq. (4.7) cancel each other. Hence, if α is a half-integer the boundary-induced part in the VEV of the charge density vanishes. Recall that the same is the case for the boundary-free part.

In a similar way, using the identity

$$\frac{g_{\beta_j}(x,y)g_{\beta_j+\epsilon_j}(x,y)}{\bar{J}_{\beta_j}^{(-)2}(x)+\bar{Y}_{\beta_j}^{(-)2}(x)}=J_{\beta_j}(y)J_{\beta_j+\epsilon_j}(y)-\frac{1}{2}\sum_{l=1,2}\frac{\bar{J}_{\beta_j}^{(-)}(x)}{\bar{H}_{\beta_j}^{(-,l)}(x)}H_{\beta_j}^{(l)}(y)H_{\beta_j+\epsilon_j}^{(l)}(y), \tag{4.11}$$

for the boundary-induced part in the VEV of the azimuthal current we find

$$\begin{aligned} \langle j^2(x)\rangle_{\mathrm{b}} &= -\frac{eq}{\pi^2 r}\sum_j\int_m^\infty dz\frac{z^2}{\sqrt{z^2-m^2}} \\ &\quad \times K_{\beta_j}(zr)K_{\beta_j+\epsilon_j}(zr)\mathrm{Re}\left[I_{\beta_j}^{(-)}(za)/K_{\beta_j}^{(-)}(za)\right]. \end{aligned} \tag{4.12}$$

As in the case of charge density, this part is a periodical function of the magnetic flux with a period equal the flux quantum.

If we present the ratio of the magnetic flux to the flux quantum in the form (3.7), then the boundary induced VEVs are functions of α_0 alone. They are odd functions of this parameter. In the limit $\alpha_0\to\pm1/2$, $|\alpha_0|<1/2$, the only nonzero contribution to $\langle j^\nu(x)\rangle_{\mathrm{b}}$ comes from the term with $j=\mp1/2$ and one has the limiting values

$$\begin{aligned} \lim_{\alpha_0\to\pm1/2}\langle j^0(x)\rangle_{\mathrm{b}} &= \pm\frac{eqm}{2\pi^2 r}K_0(2mr), \\ \lim_{\alpha_0\to\pm1/2}\langle j^2(x)\rangle_{\mathrm{b}} &= \pm\frac{eqm}{2\pi^2 r^2}K_1(2mr). \end{aligned} \tag{4.13}$$

Now comparing with Eqs. (3.34) and (3.42), we see that the limiting value of the total current density (4.4) is zero and the latter is continuous at $\alpha_0=\pm1/2$. This result was expected due to the continuity of the exterior eigenspinors as functions of the parameter α_0. Comparing with the results of the previous section, we see that the limiting transitions $a\to0$ and $|\alpha_0|\to1/2$ do not commute.

For a massless field the expressions for the boundary-induced parts in the VEVs take the form

$$
\begin{aligned}
\langle j^0(x)\rangle_{\mathrm{b}} &= \frac{eq}{2\pi^2 a^2}\sum_j \int_0^\infty dz\, \frac{K^2_{\beta_j}(zr/a) - K^2_{\beta_j+\epsilon_j}(zr/a)}{K^2_{\beta_j}(z) + K^2_{\beta_j+\epsilon_j}(z)}, \\
\langle j^2(x)\rangle_{\mathrm{b}} &= -\frac{eq}{\pi^2 a^2 r}\sum_j \int_0^\infty dz\, \frac{K_{\beta_j}(zr/a)K_{\beta_j+\epsilon_j}(zr/a)}{K^2_{\beta_j}(z) + K^2_{\beta_j+\epsilon_j}(z)} W^{(-)}_{\beta_j,\beta_j+\epsilon_j}(z),
\end{aligned} \tag{4.14}
$$

with the notation defined by Eq. (4.10). We would like to point out that the boundary-induced charge density does not vanish for a massless filed. The corresponding boundary-free part vanishes and, hence, $\langle j^0(x)\rangle = \langle j^0(x)\rangle_{\mathrm{b}}$.

Now we turn to the investigation of the boundary-induced part in the VEV of fermionic current in the asymptotic regions of the parameters. In the limit $a \to 0$, for fixed values of r, by taking into account that

$$
\frac{I^{(-)}_{\beta_j}(za)}{K^{(-)}_{\beta_j}(za)} \approx a\frac{\epsilon_j m + i\sqrt{z^2 - m^2}}{\Gamma^2(q|j+\alpha| + 1/2)}(za/2)^{2q|j+\alpha|-1}, \tag{4.15}
$$

to the leading order, from Eqs. (4.7) and (4.12), we have

$$
\begin{aligned}
\langle j^0(x)\rangle_{\mathrm{b}} &\approx \frac{eq}{\pi^2}\frac{\mathrm{sgn}(\alpha_0)(a/2r)^{2q_\alpha}}{r^2\Gamma^2(q_\alpha + 1/2)}\int_{mr}^\infty dz\, \frac{z^{2q_\alpha}}{\sqrt{z^2 - m^2 r^2}} \\
&\quad\times \left[\left(2m^2 r^2 - z^2\right) K^2_{q_\alpha - 1/2}(z) + z^2 K^2_{q_\alpha+1/2}(z)\right], \\
\langle j^2(x)\rangle_{\mathrm{b}} &\approx \frac{2eqm}{\pi^2 r^2}\frac{\mathrm{sgn}(\alpha_0)(a/2r)^{2q_\alpha}}{\Gamma^2(q_\alpha + 1/2)}\int_{mr}^\infty dz \frac{z^{2q_\alpha+1}}{\sqrt{z^2 - m^2 r^2}} K_{q_\alpha - 1/2}(z)K_{q_\alpha-1/2}(z),
\end{aligned} \tag{4.16}
$$

with the notation

$$
q_\alpha = q(1/2 - |\alpha_0|). \tag{4.17}
$$

For a massless field the asymptotic behavior for the charge density is directly obtained from Eq. (4.16). The integrals involving the Macdonald function are evaluated in terms of the gamma function and one finds

$$
\langle j^0(x)\rangle_{\mathrm{b}} \approx \frac{eq\,\mathrm{sgn}(\alpha_0)}{\pi r^2}\left(\frac{a}{2r}\right)^{2q_\alpha}\frac{q_\alpha\Gamma(2q_\alpha + 1/2)\Gamma(q_\alpha+1)}{(2q_\alpha+1)\Gamma^3(q_\alpha + 1/2)}. \tag{4.18}
$$

For the azimuthal component the leading term in Eq. (4.16) vanishes. The corresponding asymptotic behavior is directly found from Eq. (4.14). The leading term is given by

$$
\langle j^2(x)\rangle_{\mathrm{b}} \approx \frac{2eq}{\pi r^3}\frac{\mathrm{sgn}(\alpha_0)}{(2q_\alpha)^2 - 1}\left(\frac{a}{2r}\right)^{2q_\alpha+1}\frac{\Gamma(2q_\alpha + 3/2)\Gamma(q_\alpha+1)}{(2q_\alpha+1)\Gamma^3(q_\alpha + 1/2)}, \tag{4.19}
$$

for $q_\alpha > 1/2$, and by the expression

$$
\langle j^2(x)\rangle_{\mathrm{b}} \approx -\frac{eq\,\mathrm{sgn}(\alpha_0)}{2^{2q_\alpha+1}\pi r^3}\left(\frac{a}{2r}\right)^{4q_\alpha}\frac{\Gamma(1/2 - q_\alpha)}{\Gamma^4(1/2 + q_\alpha)}\Gamma(2q_\alpha + 1/2)\Gamma(3q_\alpha + 1), \tag{4.20}
$$

in the case $q_\alpha < 1/2$. For for $q_\alpha = 1/2$, the leading terms behaves as $a^2\ln(a)$.

At large distances from the boundary, for a massive field, under the condition $mr \gg 1$, the dominant contribution to the integrals come from the region near the lower limit of the integration and to the leading order we find

$$\begin{aligned} \langle j^0(x)\rangle_{\rm b} &\approx -\frac{eqm^2e^{-2rm}}{4\sqrt{\pi}(rm)^{3/2}}\sum_j \mathrm{Re}\left[I^{(-)}_{\beta_j}(ma)/K^{(-)}_{\beta_j}(ma)\right], \\ \langle j^2(x)\rangle_{\rm b} &\approx -\frac{eqm^3e^{-2rm}}{4\sqrt{\pi}(mr)^{5/2}}\sum_j \mathrm{Re}\left[I^{(-)}_{\beta_j}(ma)/K^{(-)}_{\beta_j}(ma)\right]. \end{aligned} \tag{4.21}$$

As expected, we have an exponential suppression of the boundary-induced VEVs. For a massless field, the asymptotics at large distances are given by Eqs. (4.18)-(4.20). In Fig. 4, we plot the VEVs of the charge density (left panel) and azimuthal current (right panel) for a massless fermionic field as functions of the magnetic flux. The graphs are plotted for $r/a = 1.5$ and for several values of the parameter q. As we have already mentioned, in the exterior region the total VEVs of the charge density and azimuthal current vanish in the limt $|\alpha_0| \to 0$. Note that for a massless field the boundary-free part in the VEV of the charge density vanishes and the non-zero charge density on the left plot is induced by the circular boundary.

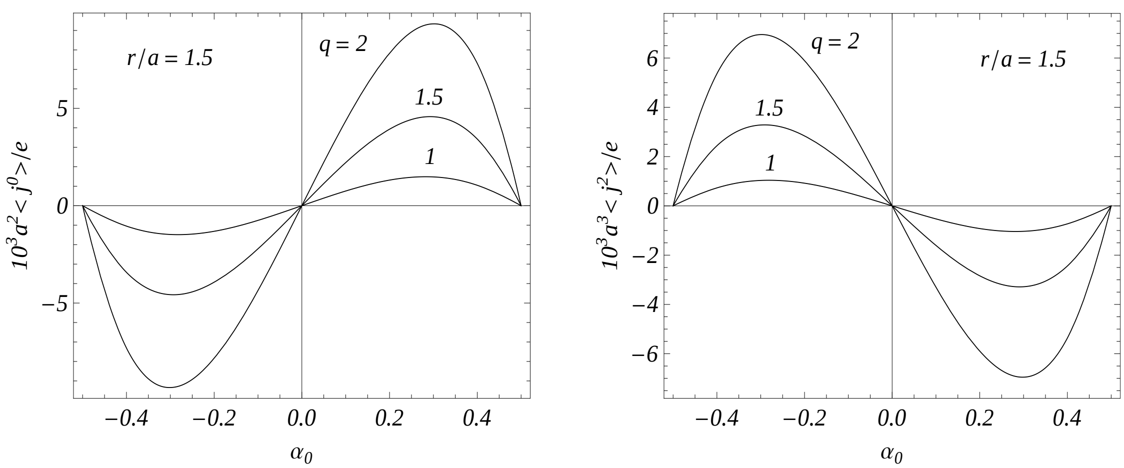

Figure 4: The VEVs of the charge density (left panel) and azimuthal current (right panel) for a massless fermionic field outside a circular boundary.

5 Fermionic current inside a circular boundary

In this section we consider the region inside a circular boundary with radius a, $r < a$, on which the fermionic field obeys the boundary condition (2.4) with $n_\mu = -\delta^1_\mu$. The boundary condition at the cone apex for the irregular mode is the same as that we have used in Section 3 for the boundary-free conical geometry. The eigenspinors in this region have the form

$$\begin{aligned} \psi^{(+)}_{\gamma j} &= \varphi_0 e^{iqj\phi - iEt}\begin{pmatrix} e^{-iq\phi/2}J_{\beta_j}(\gamma r) \\ \frac{\gamma\epsilon_j e^{iq\phi/2}}{E+m}J_{\beta_j+\epsilon_j}(\gamma r)\end{pmatrix}, \\ \psi^{(-)}_{\gamma j} &= \varphi_0 e^{-iqj\phi + iEt}\begin{pmatrix} \frac{\epsilon_j\gamma e^{-iq\phi/2}}{E+m}J_{\beta_j+\epsilon_j}(\gamma r) \\ e^{iq\phi/2}J_{\beta_j}(\gamma r)\end{pmatrix}, \end{aligned} \tag{5.1}$$

with the same notations as in Section 3. From the boundary condition at $r=a$ we find that the eigenvalues of γ are solutions of the equation

$$J_{\beta_j}(\gamma a)-\frac{\gamma\epsilon_j}{E+m}J_{\beta_j+\epsilon_j}(\gamma a)=0. \tag{5.2}$$

Note that this equation may also be written in the form $\bar{J}_{\beta_j}^{(+)}(\gamma a)=0$ with the barred notation defined by Eq. (2.20). For a given β_j, Eq. (5.2) has an infinite number of solutions which we denote by $\gamma a=\gamma_{\beta_j,l}$, $l=1,2,\ldots$.

The normalization coefficient in Eq. (5.1) is determined from the condition

$$\int_0^a dr\int_0^{\phi_0}d\phi\, r\psi_{\gamma j}^{(\pm)\dagger}\psi_{\gamma' j'}^{(\pm)}=\delta_{ll'}\delta_{jj'}\ . \tag{5.3}$$

Using the standard integral for the square of the Bessel function (see, for example, [46]), one finds

$$\varphi_0^2=\frac{a^2y^2}{\phi_0 J_{\beta_j}^2(y)}\left[2(y^2+\mu^2)-(2\epsilon_j\beta_j+1)\sqrt{y^2+\mu^2}+\mu\right]^{-1},\ y=\gamma_{\beta_j,l}, \tag{5.4}$$

where, as before, $\mu=ma$. For the further convenience we will write this expression in the form

$$\varphi_0^2=\frac{yT_{\beta_j}(y)}{2\phi_0 a^2}\frac{\mu+\sqrt{y^2+\mu^2}}{\sqrt{y^2+\mu^2}}, \tag{5.5}$$

with the notation

$$T_{\beta_j}(y)=\frac{y}{J_{\beta_j}^2(y)}\left[y^2+(\mu-\epsilon_j\beta_j)\left(\mu+\sqrt{y^2+\mu^2}\right)-\frac{y^2}{2\sqrt{y^2+\mu^2}}\right]^{-1}. \tag{5.6}$$

Substituting the eigenspinors (5.1) into the mode sum formula

$$\langle j^\nu(x)\rangle=e\sum_j\sum_{l=1}^{\infty}\bar{\psi}_{\gamma j}^{(-)}(x)\gamma^\nu\psi_{\gamma j}^{(-)}(x), \tag{5.7}$$

for the VEVs of separate components of fermionic current we have

$$\begin{aligned}\langle j^0(x)\rangle &= \frac{eq}{4\pi a^2}\sum_j\sum_{l=1}^{\infty}yT_{\beta_j}(y)\Big[\Big(\frac{\mu}{\sqrt{y^2+\mu^2}}+1\Big)J_{\beta_j}^2(yr/a)-\Big(\frac{\mu}{\sqrt{y^2+\mu^2}}-1\Big)J_{\beta_j+\epsilon_j}^2(yr/a)\Big],\\ \langle j^2(x)\rangle &= \frac{eq}{2\pi a^2 r}\sum_j\sum_{l=1}^{\infty}\frac{\epsilon_j y^2T_{\beta_j}(y)}{\sqrt{y^2+\mu^2}}J_{\beta_j}(yr/a)J_{\beta_j+\epsilon_j}(yr/a),\end{aligned} \tag{5.8}$$

with $y=\gamma_{\beta_j,l}$, and the radial component vanishes, $\langle j^1(x)\rangle=0$. As before, we assume the presence of a cutoff function without explicitly writing it.

As the explicit form for $\gamma_{\beta_j,l}$ is not known, Eqs. (5.8) are not convenient for the direct evaluation of the VEVs. In addition, the separate terms in the mode sum are highly oscillatory for large values of the quantum numbers. In order to find a convenient integral representation, we apply to the series over l the summation formula (see [48, 49])

$$\begin{aligned}\sum_{l=1}^{\infty}f(\gamma_{\beta_j,l})T_\beta(\gamma_{\beta_j,l}) &= \int_0^\infty dx\, f(x)-\frac{1}{\pi}\int_0^\infty dx\\ &\times\left[e^{-\beta_j\pi i}f(xe^{\pi i/2})\frac{K_{\beta_j}^{(+)}(x)}{I_{\beta_j}^{(+)}(x)}+e^{\beta_j\pi i}f(xe^{-\pi i/2})\frac{K_{\beta_j}^{(+)*}(x)}{I_{\beta_j}^{(+)*}(x)}\right],\end{aligned} \tag{5.9}$$

the asterisk meaning complex conjugate. Here the notation $F^{(+)}(x)$ for a given function $F(x)$ is defined by Eq. (4.8) for $x \geqslant \mu$ and by the relation

$$F^{(+)}(x) = xF'(x) + (\mu + \sqrt{\mu^2 - x^2} - \epsilon_j \beta_j)F(x), \tag{5.10}$$

for $x < \mu$. Note that in the latter case $F^{(+)*}(x) = F^{(+)}(x)$. The term in the VEVs corresponding to the first integral in the right-hand side of Eq. (5.9) coincides with the VEV of the fermionic current for the situation where the boundary is absent.

As a result, the VEV of the fermionic current is presented in the decomposed form (4.4). For the function $f(x)$ corresponding to Eq. (5.8), in the second term on the right-hand side of Eq. (5.9), the part of the integral over the region $(0, \mu)$ vanishes. Consequently, the boundary-induced contribution for the charge density in the region inside the circle is given by the expression

$$\begin{aligned} \langle j^0(x) \rangle_{\mathrm{b}} &= -\frac{eq}{2\pi^2} \sum_j \int_m^\infty dz\, z \\ &\times \Big\{ m \frac{I^2_{\beta_j}(zr) + I^2_{\beta_j+\epsilon_j}(zr)}{\sqrt{z^2 - m^2}} \mathrm{Re}[K^{(+)}_{\beta_j}(za)/I^{(+)}_{\beta_j}(za)] \\ &- [I^2_{\beta_j}(zr) - I^2_{\beta_j+\epsilon_j}(zr)] \mathrm{Im}[K^{(+)}_{\beta_j}(za)/I^{(+)}_{\beta_j}(za)] \Big\}. \end{aligned} \tag{5.11}$$

Similarly, for the boundary-induced part in the azimuthal component we find

$$\langle j^2(x) \rangle_{\mathrm{b}} = \frac{eq}{\pi^2 r} \sum_j \int_m^\infty dz\, \frac{z^2 I_{\beta_j}(zr) I_{\beta_j+\epsilon_j}(zr)}{\sqrt{z^2 - m^2}} \mathrm{Re}[K^{(+)}_{\beta_j}(za)/I^{(+)}_{\beta_j}(za)]. \tag{5.12}$$

For points away from the circular boundary, the boundary-induced contributions (5.11) and (5.12) are finite and the renormalization is reduced to that for the boundary-free geometry. These contributions are periodic functions of the parameter α with the period equal to 1. So, if we present this parameter in the form (3.7) with n_0 being an integer, then the VEVs depend on α_0 alone and they are odd functions of this parameter. Similar to Eq. (4.9), the ratio of the combinations of the modified Bessel functions in Eq. (5.11) is presented in the form

$$\frac{K^{(+)}_{\beta_j}(x)}{I^{(+)}_{\beta_j}(x)} = \frac{W^{(+)}_{\beta_j,\beta_j+\epsilon_j}(x) + i\sqrt{1 - \mu^2/x^2}}{x[I^2_{\beta_j}(x) + I^2_{\beta_j+\epsilon_j}(x)] + 2\mu I_{\beta_j}(x) I_{\beta_j+\epsilon_j}(x)}, \tag{5.13}$$

with the notation defined by Eq. (4.10).

When the parameter α is a half-integer the contributions of the modes with $j \neq -\alpha$ to the boundary-induced VEVs inside the circle are still given by expressions (5.11) and (5.12). It can be easily seen that the contributions of the modes with $j < -\alpha$ and $j > -\alpha$ cancel each other. The contribution of the mode with $j = -\alpha$ should be considered separately. In Appendix C we show that this contribution vanishes as well. Therefore, we conclude that for α being a half-integer, the boundary-induced part in the VEV of the fermionic current vanishes.

For a massless field the formulae of the boundary-induced parts are simplified to

$$\begin{aligned} \langle j^0(x) \rangle_{\mathrm{b}} &= \frac{eq}{2\pi^2 a^2} \sum_j \int_0^\infty dz\, \frac{I^2_{\beta_j}(zr/a) - I^2_{\beta_j+\epsilon_j}(zr/a)}{I^2_{\beta_j}(z) + I^2_{\beta_j+\epsilon_j}(z)}, \\ \langle j^2(x) \rangle_{\mathrm{b}} &= \frac{eq}{\pi^2 r a^2} \sum_j \int_0^\infty dz\, \frac{I_{\beta_j}(zr/a) I_{\beta_j+\epsilon_j}(zr/a)}{I^2_{\beta_j}(z) + I^2_{\beta_j+\epsilon_j}(z)} W^{(+)}_{\beta_j,\beta_j+\epsilon_j}(z). \end{aligned} \tag{5.14}$$

Note that for a massless field $W^{(+)}_{\beta_j,\beta_j+\epsilon_j}(z) = W^{(-)}_{\beta_j,\beta_j+\epsilon_j}(z)$. In the limit $\alpha_0 \to \pm 1/2$, $|\alpha_0| < 1/2$, the only nonzero contributions to Eqs. (5.11) and (5.12) come from the term with $j = \mp 1/2$ and, by making use of Eqs. (3.34), for the total VEVs we find

$$\begin{aligned}
\lim_{\alpha_0 \to \pm 1/2} \langle j^0(x) \rangle &= \mp \frac{eqm}{2\pi^2 r} K_0(2mr) \pm \frac{eq}{\pi^2 r} \int_m^\infty dz \, \frac{amz\cosh(2zr) - (z+m)e^{2za}}{\sqrt{z^2 - m^2}(\frac{z+m}{z-m}e^{4za} + 1)}, \\
\lim_{\alpha_0 \to \pm 1/2} \langle j^2(x) \rangle &= \mp \frac{eqm}{2\pi^2 r^2} K_1(2mr) \mp \frac{eqa}{\pi^2 r^2} \int_m^\infty dz \, \frac{z^2}{\sqrt{z^2 - m^2}} \frac{\sinh(2zr)}{\frac{z+m}{z-m}e^{4za} + 1}.
\end{aligned} \tag{5.15}$$

For a massless field they reduce to the expressions:

$$\begin{aligned}
\lim_{\alpha_0 \to \pm 1/2} \langle j^0(x) \rangle &= \mp \frac{eq}{8a\pi r}, \\
\lim_{\alpha_0 \to \pm 1/2} \langle j^2(x) \rangle &= \mp \frac{eq}{8\pi^2 r^3} \left[1 + \frac{\pi r/2a}{\sin(\pi r/2a)} \right].
\end{aligned} \tag{5.16}$$

Note that the limiting values are linear functions of the parameter q.

The general expressions for the VEVs are simplified in asymptotic regions of the parameters. First we consider large values of the circle radius. For the modified Bessel functions in the integrands of Eqs. (5.11) and (5.12), with za in their arguments, we use the asymptotic expansions for large values of the argument. By taking into account that for a massive field the dominant contribution into the integrals comes from the integration range near the lower limit, to the leading order we find

$$\begin{aligned}
\langle j^0(x) \rangle_{\rm b} &\approx \frac{eqm^2 e^{-2ma}}{8\sqrt{\pi}(ma)^{3/2}} \sum_j \epsilon_j \left[(\beta_j + \epsilon_j) I^2_{\beta_j}(mr) + \beta_j I^2_{\beta_j+\epsilon_j}(mr) \right], \\
\langle j^2(x) \rangle_{\rm b} &\approx -\frac{eqm^2 e^{-2ma}}{8\sqrt{\pi} r (ma)^{3/2}} \sum_j (2\epsilon_j \beta_j + 1) I_{\beta_j}(mr) I_{\beta_j+\epsilon_j}(mr).
\end{aligned} \tag{5.17}$$

In this limit, for a fixed value of the radial coordinate, the boundary-induced VEVs decay exponentially. For a massless field, assuming $r/a \ll 1$, we expand the modified Bessel function in the numerators of integrands in Eq. (5.14) in powers of r/a. The dominant contribution comes from the term $j = 1/2$ for $\alpha_0 < 0$ and from the term $j = -1/2$ for $\alpha_0 > 0$. To the leading order we have

$$\begin{aligned}
\langle j^0(x) \rangle_{\rm b} &\approx -\frac{eq}{2\pi^2 a^2} \frac{\mathrm{sgn}(\alpha_0)(r/2a)^{2q_\alpha - 1}}{\Gamma^2(q_\alpha + 1/2)} \int_0^\infty dz \, \frac{z^{2q_\alpha - 1}}{I^2_{q_\alpha+1/2}(z) + I^2_{q_\alpha-1/2}(z)}, \\
\langle j^2(x) \rangle_{\rm b} &\approx -\frac{eq}{\pi^2 a^3} \frac{\mathrm{sgn}(\alpha_0)(r/2a)^{2q_\alpha - 1}}{(2q_\alpha + 1)\Gamma^2(q_\alpha + 1/2)} \int_0^\infty dz \, \frac{z^{2q_\alpha} W^{(+)}_{q_\alpha-1/2,q_\alpha+1/2}(z)}{I^2_{q_\alpha+1/2}(z) + I^2_{q_\alpha-1/2}(z)},
\end{aligned} \tag{5.18}$$

where q_α is defined in Eq. (4.17). As it is seen, for a massless field the decay of the VEVs is as power-law.

For points near the apex of the cone, $r \to 0$, we have the following leading terms

$$\langle j^0(x)\rangle_{\rm b} \approx \frac{eq}{2\pi^2 a^2}\frac{\mathrm{sgn}(\alpha_0)(r/2a)^{2q_\alpha-1}}{\Gamma^2(q_\alpha+1/2)}\int_\mu^\infty dz\,\frac{z^{2q_\alpha}}{\sqrt{z^2-\mu^2}}$$
$$\times\frac{\mu W^{(+)}_{q_\alpha-1/2,q_\alpha+1/2}(z)-(z^2-\mu^2)/z}{z[I^2_{q_\alpha-1/2}(z)+I^2_{q_\alpha+1/2}(z)]+2\mu I_{q_\alpha-1/2}(z)I_{q_\alpha+1/2}(z)}, \tag{5.19}$$

$$\langle j^2(x)\rangle_{\rm b} \approx -\frac{eq}{\pi^2 a^3}\frac{\mathrm{sgn}(\alpha_0)(r/2a)^{2q_\alpha-1}}{(2q_\alpha+1)\Gamma^2(q_\alpha+1/2)}\int_\mu^\infty dz\,\frac{z^{2q_\alpha+2}}{\sqrt{z^2-\mu^2}}$$
$$\times\frac{W^{(+)}_{q_\alpha-1/2,q_\alpha+1/2}(z)}{z[I^2_{q_\alpha-1/2}(z)+I^2_{q_\alpha+1/2}(z)]+2\mu I_{q_\alpha-1/2}(z)I_{q_\alpha+1/2}(z)}. \tag{5.20}$$

For a massless field these expressions reduce to Eq. (5.18). From here it follows that in the limit $r \to 0$ the boundary-induced part vanishes when $|\alpha_0| < 1/2 - 1/(2q)$ and diverges for $|\alpha_0| > 1/2 - 1/(2q)$. Notice that in the former case the irregular mode is absent and the divergence in the latter case comes from the irregular mode. This divergence is integrable. In the case $|\alpha_0| = 1/2 - 1/(2q)$, corresponding to $q_\alpha = 1/2$, the boundary-induced VEV tends to a finite limiting value. In particular, for the magnetic vortex in the background Minkowski spacetime, the boundary-induced contribution diverges as $r^{-2|\alpha_0|}$. In Fig. 5, the VEVs of the charge density (left panel) and azimuthal current (right panel) are plotted for a massless fermionic field inside a circular boundary as functions of the magnetic flux. The graphs are plotted for $r/a = 0.5$ and for several values of the opening angle for the conical space. For a massless field the boundary-free part in the VEV of the charge density vanishes and the charge density on the left plot is induced by the boundary.

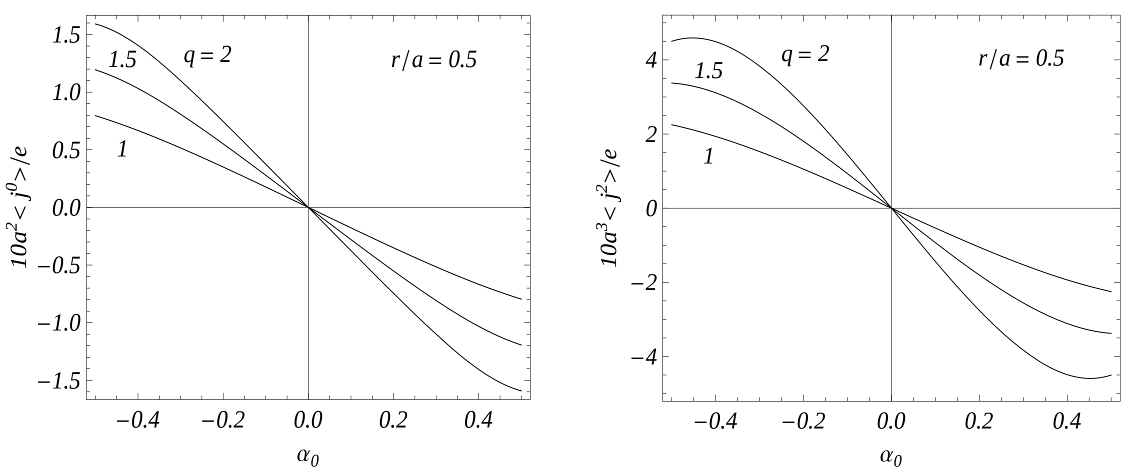

Figure 5: The same as in figure 4 for the region inside a circular boundary.

6 Summary and conclusions

In this paper we have investigated the VEV of the fermionic current induced by a magnetic flux string in a (2+1)-dimensional conical spacetime with a circular boundary. The case of massive fermionic field is considered with MIT bag boundary condition on the circle. In (2+1)-dimensional spacetime there are two inequivalent irreducible representations of the Dirac matri-

ces. We have used the representation (2.5). The corresponding results for the second representation are obtained by changing $m \rightarrow -m$. Under this change, the boundary-free contribution to the VEV of the azimuthal current is not changed, whereas the VEV of the charge density changes the sign. For the evaluation of the expectation values we have employed the direct mode summation method. The corresponding positive and negative energy eigenspinors in the region outside the circular boundary are given by Eqs. (2.21) and (2.27).

The VEV of the fermionic current in the boundary-free conical space is investigated in Sect. 3. For this geometry, under the condition (3.8), there are no square integrable irregular modes. In the case $|\alpha_0| > (1 - 1/q)/2$, the theory of von Neumann deficiency indices leads to a one-parameter family of allowed boundary conditions at the origin. Here we consider a special boundary condition that arises when imposing bag boundary condition at a finite radius, which is then shrunk to zero. The VEVs of the fermionic current for other boundary conditions on the cone apex are evaluated in a similar way. The contribution of the regular modes is the same for all boundary conditions and the formulae differ by the parts related to the irregular modes. For the boundary condition under consideration, the eigenspinors for the boundary-free geometry are obtained from the corresponding functions in the region outside a circular boundary with radius a, taking the limit $a \rightarrow 0$. They are presented by Eq. (3.1). When the magnetic flux, measured in units of the flux quantum, is a half-integer, there is a special mode corresponding to the angular momentum $j = -\alpha$, with the negative-energy eigenspinor given by Eq. (3.6).

In the boundary-free geometry, the regularized expressions of the VEVs are given by Eqs. (3.14) and (3.17) for the charge density and the azimuthal current, respectively, and the VEV of the radial component vanishes. These VEVs are periodic functions of the parameter α with the period equal to 1. So, if we present this parameter as (3.7), with n_0 being an integer number, then the VEVs are functions of α_0 alone. These functions are odd with respect to the reflection $\alpha_0 \rightarrow -\alpha_0$. Both charge density and azimuthal current exhibit the jump structure at half-integer values for the ratio of the magnetic flux to the flux quantum. Simple expressions for the renormalized VEVs, Eqs. (3.26) and (3.28), are obtained in the special case where the parameter q related to the planar angle deficit is an integer and the magnetic flux takes special values given by Eq. (3.23). In the general case of parameters α and q, we have derived two different representations for the renormalized VEVs. The first one is based on Eq. (A.6) and the corresponding expressions for the charge density and azimuthal current have the forms (3.31) and (3.37). The second representation is obtained by using the Abel-Plana summation formula and the corresponding expressions are given by Eqs. (3.43), (3.46). For a massless field the VEV of the charge density vanishes for points outside the magnetic vortex and the VEV of the azimuthal current behaves as r^{-3}. For a massive field, for points near the vortex the VEVs behave as $1/r$ and $1/r^3$ for the charge density and the azimuthal current, respectively. At distances larger than the fermion Compton wavelength, the VEVs decay exponentially with the decay rate depending on the opening angle of the cone. The total charge induced by the magnetic vortex does not depend on the angle deficit of the conical space and is given by Eq. (3.40). In the special case of a magnetic flux in Minkowski spacetime, the formulae for the VEVs of the charge density and the azimuthal current reduce to Eqs. (3.41), or equivalently, Eqs. (3.47). In Appendix B we show the equivalence of Eqs. (3.47) to the expressions for the VEVs of charge density and azimuthal current known from the literature.

The effects of a circular boundary on the VEV of the fermionic current are considered in Sect. 4 for the exterior region. From the point of view of the physics in this region, the circular boundary can be considered as a simple model for the defect core. The mode sums of the charge density and the azimuthal current are given by Eqs. (4.3) and the radial component vanishes. In order to extract from the VEVs the contributions induced by the boundary, we have subtracted the boundary-free parts. Rotating the integration contours in the complex plane, we have derived

rapidly convergent integral representations for the boundary-induced contributions, Eqs. (4.7) and (4.12). These formulae are simplified in the case of a massless field with expressions (4.14). In the exterior region, the total VEV of the fermionic current is a continuous function of the magnetic flux. Note that unlike the boundary-free part, the boundary-induced part in the VEV of the charge density for a massless field is not zero. The parts in the VEVs induced by the boundary are periodic functions of the magnetic flux with the period equal to the flux quantum. These parts vanish for the special case of the magnetic flux corresponding to $|\alpha_0| = 1/2$. In the limit when the radius of the circle goes to zero and for $|\alpha_0| < 1/2$, for a massive field the boundary-induced contributions in the exterior region behave as a^{2q_α}, with q_α defined by Eq. (4.17). For a massless field the corresponding asymptotics are given by Eqs. (4.18)-(4.20). At large distances from the boundary and for a massive field, the contributions coming from the boundary decay exponentially [see Eqs. (4.21)]. In the same limit and for a massless field, the boundary-induced VEV in the charge density decays as $(a/r)^{2q_\alpha}$. For the azimuthal current, the contribution induced by a circular boundary behaves as $(a/r)^{2q_\alpha+1}$ when $q_\alpha > 1/2$, and like $(a/r)^{4q_\alpha}$ for $q_\alpha < 1/2$. Note that, when the circular boundary is present, the VEVs of physical quantities in the exterior region are uniquely defined by the boundary conditions and by the bulk geometry. This means that if we consider a non-trivial core model for both conical space and magnetic flux with finite thickness $b < a$ and with the line element (2.1) in the region $r > b$, the results in the region outside the circular boundary will not be changed.

The boundary-induced VEVs in the region inside a circular boundary are studied in Sect. 5. The corresponding mode sums for the charge density and the azimuthal current are given by Eq. (5.8). They contain series over the zeros of the function given by Eq. (5.2). For the summation of these series we have employed a variant of the generalized Abel-Plana formula. The latter allowed us to extract explicitly from the VEVs the parts corresponding to the conical space without boundaries and to present the contributions induced by the circle in terms of exponentially convergent integrals. In the interior region the boundary-induced parts in the renormalized VEVs of the charge density and the azimuthal current are given by Eqs. (5.11) and (5.12). For a massless fermionic field these formulae are reduced to Eqs. (5.14). For large values of the circle radius and for a massive field the boundary-induced contribution decay exponentially [Eqs. (5.17)]. In the case of a massless field, the VEVs decay as $(r/a)^{2q_\alpha-1}$, for both charge density and azimuthal current. For points near the apex of the cone, the leading terms in the corresponding asymptotic expansions are given by Eqs. (5.19) and (5.20). In particular, the boundary-induced parts vanish at the apex when $|\alpha_0| < 1/2-1/(2q)$ and diverge for $|\alpha_0| > 1/2 - 1/(2q)$.

The formulas for the VEV of the fermionic current are easily generalized for the case of a spinor field with quasiperiodic boundary condition (2.28) along the azimuthal direction. This problem is reduced to the one we have considered by a gauge transformation. The corresponding expressions for the VEVs are obtained from those given above changing the definition of the parameter α by Eq. (2.29).

The results obtained in the present paper can be applied for the evaluation of the VEV of the fermionic current in graphitic cones. Graphitic cones are obtained from graphene sheet if one or more sectors are excised. The opening angle of the cone is related to the number of sectors removed, N_c, by the formula: $\phi_0 = 2\pi(1 - N_c/6)$, with $N_c = 1, 2, \ldots, 5$ (for the electronic properties of graphitic cones see, e.g., [50] and references therein). All these angles have been observed in experiments [51]. The electronic band structure of graphene close to the Dirac points shows a conical dispersion $E(\mathbf{k}) = v_F|\mathbf{k}|$, where $\mathbf{k}$ is the momentum measured relatively to the Dirac points and $v_F \approx 10^8$ cm/s represents the Fermi velocity which plays the role of a speed of light. Consequently, the long-wavelength description of the electronic states in graphene can be formulated in terms of the Dirac-like theory in (2+1)-dimensional spacetime. The corresponding

excitations are described by a pair of two-component spinors, corresponding to the two different triangular sublattices of the honeycomb lattice of graphene (see, for instance, [40]). In both cases of finite and truncated graphitic cones the corresponding 2-dimensional surface has a circular boundary. As the Dirac field lives on the cone surface, it is natural to impose bag boundary condition (2.4) on the bounding circle which ensures the zero fermion flux through the edge of the cone. A more detailed investigation of the fermionic current in graphitic cones, based on the results of the present paper, will be presented elsewhere.

Acknowledgments

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A Integral representations

In this section we derive two integral representations for the function $\mathcal{I}(q,\alpha_0,z)$ defined by Eq. (3.19). In the first approach, we use the integral representation for the modified Bessel function $I_{\beta_j}(z)$ (see formula 9.6.20 in Ref. [52]). To be allowed to replace the order of the integration and the summation over j, we integrate by parts the first term in this representation:

$$I_{\beta_j}(z) = \frac{\sin(\pi\beta_j)}{\pi\beta_j}e^{-x} + \frac{z}{\pi}\int_0^{\pi} dy \sin y \frac{\sin(\beta_j y)}{\beta_j} e^{z\cos y} - \frac{\sin(\pi\beta_j)}{\pi}\int_0^{\infty} dy e^{-z\cosh y - \beta_j y}. \tag{A.1}$$

Substituting (A.1) into Eq. (3.19) and interchanging the order of the summation and integration, we apply the formula [46]

$$\sum_j \frac{\sin(\beta_j y)}{\beta_j} = (-1)^l \frac{\pi\cos\left[(2l+1)\pi(\alpha_0 - 1/2q)\right]}{q\cos[\pi(\alpha_0 - 1/2q)]}, \tag{A.2}$$

for $2l\pi/q < y < (2l+2)\pi/q$. For the first term in the right-hand side of (A.1) one has $y = \pi$. For this term, when $q = 2l$ with $l = 1, 2, \ldots$, the corresponding summation formula has the form

$$\sum_j \frac{\sin(\pi\beta_j)}{\beta_j} = (-1)^{q/2}\frac{\pi}{q}\cos(q\pi\alpha_0)\tan[\pi(\alpha_0 - 1/2q)]. \tag{A.3}$$

Finally, for the series corresponding to the last term in Eq. (A.1) we have

$$\sum_j \sin(\pi\beta_j) e^{-\beta_j y} = \frac{f(q,\alpha_0,y)}{\cosh(qy) - \cos(q\pi)}, \tag{A.4}$$

with the notation

$$\begin{aligned} f(q,\alpha_0,y) &= \cos\left[q\pi\left(1/2-\alpha_0\right)\right]\cosh\left[\left(q\alpha_0 + q/2 - 1/2\right)y\right] \\ &\quad - \cos\left[q\pi\left(1/2+\alpha_0\right)\right]\cosh\left[\left(q\alpha_0 - q/2 - 1/2\right)y\right]. \end{aligned} \tag{A.5}$$

Combining the formulae given above, we find the following integral representation for the series (3.19):

$$\begin{aligned} \mathcal{I}(q,\alpha_0,z) &= \frac{e^z}{q} - \frac{1}{\pi}\int_0^{\infty} dy \frac{e^{-z\cosh y} f(q,\alpha_0,y)}{\cosh(qy) - \cos(q\pi)} \\ &\quad + \frac{2}{q}\sum_{l=1}^{p}(-1)^l \cos[2\pi l(\alpha_0 - 1/2q)] e^{z\cos(2\pi l/q)}, \end{aligned} \tag{A.6}$$

with $2p < q < 2p+2$. In the case $q = 2p$, the term

$$-(-1)^{q/2}\frac{e^{-z}}{q}\sin(q\pi\alpha_0), \tag{A.7}$$

should be added to the right-hand side of Eq. (A.6). Note that for $1 \leqslant q < 2$, the last term on the right-hand side of Eq. (A.6) is absent. Formula (A.6) is simplified in the case $q = 1$:

$$\mathcal{I}(1,\alpha_0,z) = e^z - \frac{\sin(\pi\alpha_0)}{\pi}\int_0^\infty dy e^{-z\cosh y}\frac{\cosh[(1/2-\alpha_0)\,y]}{\cosh(y/2)}. \tag{A.8}$$

In the special case (3.23) with an integer q, the integral term in Eq. (A.6) vanishes. For $q = 2p+1$ one finds $\mathcal{I}(q,\alpha,z) = (2/q)\sum_{l=0}^{(q-1)/2\prime} e^{z\cos(2\pi l/q)}$, where, as before, the prime on the summation sign mens that the term with $l = 0$ should be halved. For even values of q, by taking into account the additional term (A.7), we find $\mathcal{I}(q,\alpha,z) = (2/q)\sum_{l=0}^{q/2\prime} e^{z\cos(2\pi l/q)} - e^{-z}/q$. Note that for an integer q, from definition (3.19) one has $\mathcal{I}(q,\alpha,z) = 2\sum_{n=0}^{\infty\prime} I_{qn}(z)$. Now, we can see that in the special case under consideration, formula (A.6) coincides with Eq. (3.24).

We can give an alternative integral representation of the series (3.19) by using the Abel-Plana summation formula in the form (see [27, 49])

$$\sum_{n=0}^{\infty} f(n+1/2) = \int_0^\infty dz\, f(z) - i\int_0^\infty dz\,\frac{f(iz)-f(-iz)}{e^{2\pi z}+1}. \tag{A.9}$$

First of all we write this formula in the form more appropriate for the application to Eq. (3.19). Let us consider the series

$$\sum_j f(|j+u| + v\epsilon_j) = \sum_{\delta=\pm1}\sum_{n=0}^{\infty} f(n+1/2+\delta|u+v|), \tag{A.10}$$

where $|u| \leqslant 1/2$, $|v| \leqslant 1/2$. Applying formula (A.9), we get

$$\sum_j f(|j+u| + v\epsilon_j) = \sum_{\delta=\pm1}\int_0^\infty dz\, f(z+\delta w) - i\sum_{\delta=\pm1}\int_0^\infty dz\,\frac{\delta f(i\delta(z+iw)) + \delta f(i\delta(z-iw)}{e^{2\pi z}+1}, \tag{A.11}$$

with the notation $w = |u+v|$. Introducing new integration variables, we present this formula in the form

$$\begin{aligned}\sum_j f(|j+u| + v\epsilon_j) &= 2\int_0^\infty dz f(z) - \int_0^w dz\,[f(z) - f(-z)] \\ &\quad -i\sum_{\delta=\pm1}\int_{i\delta w}^{\infty+i\delta w} dz\,\frac{f(iz)-f(-iz)}{e^{2\pi(z-i\delta w)}+1}.\end{aligned} \tag{A.12}$$

Now, deforming the integration contour, we write the integral along the half-line $(i\delta w, \infty + i\delta w)$ in the complex plane z as the sum of the integrals along the segment $(i\delta w, 0)$ and along $(0,\infty)$. At this step we note that in the case $1/2 < w < 1$ the integrand has a pole at $z = i\delta(w-1/2)$. We exclude the poles by small semicircles in the right-half plane with radius tending to zero (see Fig. 6). The sum of the integrals along the segments $(iw, 0)$ and $(-iw, 0)$ cancels the second integral in the right-hand side of Eq. (A.12) and we get the following result

$$\sum_j f(|j+u| + v\epsilon_j) = A + 2\int_0^\infty dx f(x) - i\int_0^\infty dy\sum_{\delta=\pm1}\frac{f(iy)-f(-iy)}{e^{2\pi(y+i\delta|u+v|)}+1}, \tag{A.13}$$

where $A=0$ for $0 \leqslant |u+v| \leqslant 1/2$, and

$$A = f(1/2 - |u+v|) - f(|u+v| - 1/2), \tag{A.14}$$

for $1/2 < |u+v| < 1$. The term A comes from the contributions of the above mentioned poles to the integrals.

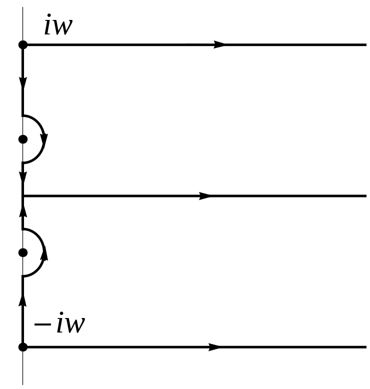

Figure 6: Integration contour in the complex plane z in the derivation of summation formula (A.13).

We apply to the series in Eq. (3.19) formula (A.13) with the function $f(z) = I_{qz}(x)$ and with the parameters $u = \alpha_0$, $v = -1/(2q)$. This leads to the following result

$$\begin{aligned} \mathcal{I}(q, \alpha_0, x) &= A(q, \alpha_0, x) + \frac{2}{q}\int_0^\infty dz\, I_z(x) \\ &- \frac{4}{\pi q}\int_0^\infty dz\, \mathrm{Re}\left[\frac{\sinh(z\pi) K_{iz}(x)}{e^{2\pi(z+i|q\alpha_0-1/2|)/q}+1}\right], \end{aligned} \tag{A.15}$$

with $A(q, \alpha_0, x) = 0$ for $|\alpha_0 - 1/2q| \leqslant 1/2$, and

$$A(q, \alpha_0, x) = \frac{2}{\pi}\sin[\pi(|q\alpha_0 - 1/2| - q/2)] K_{|q\alpha_0-1/2|-q/2}(x), \tag{A.16}$$

for $1/2 < |\alpha_0 - 1/2q| < 1$. Note that in Eq. (A.15) the function $K_{iz}(x)$ is real and the real part in the integrand can be written explicitly by observing that

$$\mathrm{Re}\,(e^{u+iv}+1)^{-1} = \frac{1}{2}\frac{e^{-u}+\cos v}{\cosh u + \cos v}. \tag{A.17}$$

In the special case $q = 1$, from (A.15) we have

$$\begin{aligned} \mathcal{I}(1, \alpha_0, x) &= A(1, \alpha_0, x) + 2\int_0^\infty dz\, I_z(x) \\ &+ \frac{4}{\pi}\int_0^\infty dz\, \mathrm{Re}\left[\frac{\sinh(z\pi) K_{iz}(x)}{e^{2\pi(z-i\alpha_0)}-1}\right], \end{aligned} \tag{A.18}$$

with $A(1, \alpha_0, x) = -(2/\pi)\sin(\pi\alpha_0) K_{\alpha_0}(x)$ in the case $-1/2 < \alpha_0 < 0$ and $A(1, \alpha_0, x) = 0$ for $0 \leqslant \alpha_0 \leqslant 1/2$. Note that the last term in the right-hand side is an even function of α_0.

B Comparison with the previous results in the Minkowski bulk

The VEVs of the fermionic current induced by a magnetic flux in (2+1)-dimensional Minkowski spacetime have been previously investigated in Ref. [35]. The expressions for the VEVs of charge density and the azimuthal current derived in this paper have the form (in our notations)

$$\begin{aligned}
\langle j^0(x)\rangle_{0,\mathrm{ren}} &= -em\frac{\sin(\pi\alpha_0)}{\pi^3}\int_m^\infty dk\frac{kK_{\alpha_0}^2(kr)}{\sqrt{k^2-m^2}},\\
\langle j^2(x)\rangle_{0,\mathrm{ren}} &= e\frac{\sin(\pi\alpha_0)}{\pi^3}\int_m^\infty dk\,k^3\frac{K_{\alpha_0}^2(kr)-K_{\alpha_0-1}(kr)K_{\alpha_0+1}(kr)}{\sqrt{k^2-m^2}}.
\end{aligned} \tag{B.1}$$

In this section, we show that these representations are equivalent to Eqs. (3.47). First we consider the charge density. As the first step, we use the integral representation for the square of the Macdonald function:

$$K_{\alpha_0}^2(x) = \frac{1}{2}\int_0^\infty \frac{dz}{z}e^{-x^2/2z-z}K_{\alpha_0}(z). \tag{B.2}$$

This formula is easily obtained from the integral representation of the product of two Macdonald functions given in Ref. [53] (page 439). Inserting Eq. (B.2) into (B.1), after the change of the order of integrations, the integral over k is taken simply and for the charge density we obtain the result given in Eq. (3.47).

In order to see the equivalence of the representations for the azimuthal current, we present the expression with the Macdonald functions in the form

$$K_{\alpha_0}^2(kr) - K_{\alpha_0-1}(kr)K_{\alpha_0+1}(kr) = -\frac{2}{r^2}\int_r^\infty dx x K_{\alpha_0}^2(kx). \tag{B.3}$$

After substituting this into Eq. (B.1), we use the integral representation (B.2). Then, we first take the integral over k and after that the integral over x. As a result, the integral representation (3.47) for the azimuthal current is obtained. Hence, we have shown that, in the special case of Minkowski bulk, our results for the VEVs of the charge density and azimuthal current, given by Eq. (3.47), agree with those from the literature. Note that we have also derived alternative representations (3.41). For the numerical evaluation the latter are more convenient.

C Contribution of the mode with $j = -\alpha$

When the parameter α is equal to a half-integer, the contribution of the mode with $j = -\alpha$ to the VEV of the fermionic current should be evaluated separately. Here we consider the region inside a circle with radius a. Similar to Eq. (3.6), the negative-energy eigenspinor for this mode has the form

$$\psi_{\gamma,-\alpha}^{(-)}(x) = \frac{b_0}{\sqrt{r}}e^{iq\alpha\phi+iEt}\begin{pmatrix} \frac{\gamma e^{-iq\phi/2}}{E+m}\sin(\gamma r-\gamma_0) \\ e^{iq\phi/2}\cos(\gamma r-\gamma_0)\end{pmatrix}, \tag{C.1}$$

where γ_0 is defined after Eq. (3.6). From boundary condition (2.4) it follows that the eigenvalues of γ are solutions of the equation

$$m\sin(\gamma a) + \gamma\cos(\gamma a) = 0. \tag{C.2}$$

The positive roots of this equation we denote by $\gamma_l = \gamma a$, $l = 1, 2, \ldots$. From the normalization condition, for the coefficient in (C.1) one has

$$b_0^2 = \frac{E+m}{aE\phi_0}\left[1-\sin(2\gamma a)/(2\gamma a)\right]^{-1}. \tag{C.3}$$

Using Eq. (C.1), for the contributions of the mode under consideration to the VEVs of the fermionic current we find:

$$\begin{aligned}\langle j^0(x)\rangle_{j=-\alpha} &= \frac{e}{ar\phi_0}\sum_{l=1}^{\infty}\frac{1+\mu\left[\gamma_l\sin(2\gamma_l r/a)-\mu\cos(2\gamma_l r/a)\right]/(aE)^2}{1-\sin(2\gamma_l)/(2\gamma_l)},\\ \langle j^2(x)\rangle_{j=-\alpha} &= -\frac{e}{ar^2\phi_0}\sum_{l=1}^{\infty}\frac{\gamma_l}{(aE)^2}\frac{\mu\sin(2\gamma_l r/a)+\gamma_l\cos(2\gamma_l r/a)}{1-\sin(2\gamma_l)/(2\gamma_l)},\end{aligned}\tag{C.4}$$

where $(aE)^2=\gamma_l^2+\mu^2$ and $\mu=ma$. The part corresponding to the radial component vanishes. We assume the presence of a cutoff function. For the summation of the series in Eqs. (C.4), we use the Abel-Plana-type formula

$$\sum_{l=1}^{\infty}\frac{\pi f(\gamma_l)}{1-\sin(2\gamma_l)/(2\gamma_l)}=-\frac{\pi f(0)/2}{1/\mu+1}+\int_0^{\infty}dz\,f(z)-i\int_0^{\infty}dz\frac{f(iz)-f(-iz)}{\frac{z+\mu}{z-\mu}e^{2z}+1}.\tag{C.5}$$

Eq. (C.5) is obtained from more general summation formula given in [54] (see also [49]) taking $b_1=0$ and $b_2=-1/\mu$. For the functions $f(z)$ corresponding to Eq. (C.4) one has $f(0)=0$. The contribution of the last term in Eq. (C.5) to $\langle j^{\nu}(x)\rangle_{j=-\alpha}$ is finite in the limit when the cutoff is removed. Noting that for both series in Eq. (C.4) the function $f(z)$ is an even function, we conclude that the last term in (C.5) does not contribute to the both charge density and azimuthal current. For the remaining parts coming from the second term in the right-hand side of Eq. (C.5) one has

$$\begin{aligned}\langle j^0(x)\rangle_{j=-\alpha} &= \frac{e}{\pi r\phi_0}\int_0^{\infty}d\gamma\left[1+h(\gamma)\right],\\ \langle j^2(x)\rangle_{j=-\alpha} &= -\frac{e}{\pi r^2\phi_0}\int_0^{\infty}d\gamma\left[\cos(2\gamma r)+h(\gamma)\right],\end{aligned}\tag{C.6}$$

with $h(\gamma)=mE^{-2}\left[\gamma\sin(2\gamma r)-m\cos(2\gamma r)\right]$ and $E^2=\gamma^2+m^2$. These parts do not depend on the circle radius. Now, we can see that the integral with the function $h(\gamma)$ is zero. The remained term in the charge density is subtracted by the renormalization and the remaining integral in the expression of azimuthal current is zero for $r>0$. Hence, we conclude that the term with $j=-\alpha$ does not contribute to the renormalized VEV of the fermionic current.

References

[1] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).

[2] S. Sarangi and S.H.H. Tye, Phys. Lett. B **536**, 185 (2002); E.J. Copeland, R.C. Myers, and J. Polchinski, JHEP **06**, 013 (2004); G. Dvali and A. Vilenkin, JCAP **0403**, 010 (2004).

[3] T.M. Helliwell and D.A. Konkowski, Phys. Rev. D **34**, 1918 (1986).

[4] W.A. Hiscock, Phys. Lett. B **188**, 317 (1987).

[5] B. Linet, Phys. Rev. D **35**, 536 (1987).

[6] V.P. Frolov and E.M. Serebriany, Phys. Rev. D **35**, 3779 (1987).

[7] J.S. Dowker, Phys. Rev. D **36**, 3095 (1987); J.S. Dowker, Phys. Rev. D **36**, 3742 (1987).

[8] P.C.W. Davies and V. Sahni, Class. Quantum Grav. **5**, 1 (1988).

[9] A.G. Smith, in *The Formation and Evolution of Cosmic Strings*, Proceedings of the Cambridge Workshop, Cambridge, England, 1989, edited by G.W. Gibbons, S.W. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, England, 1990).

[10] G.E.A. Matsas, Phys. Rev. D **41**, 3846 (1990).

[11] B. Allen and A.C. Ottewill, Phys. Rev. D **42**, 2669 (1990); B. Allen, J.G. Mc Laughlin, and A.C. Ottewill, Phys. Rev. D **45**, 4486 (1992);B. Allen, B.S. Kay, and A.C. Ottewill, Phys. Rev. D **53**, 6829 (1996).

[12] T. Souradeep and V. Sahni, Phys. Rev. D **46**, 1616 (1992).

[13] K. Shiraishi and S. Hirenzaki, Class. Quantum Grav. **9**, 2277 (1992).

[14] V.B. Bezerra and E.R. Bezerra de Mello, Class. Quantum Grav. **11**, 457 (1994); E.R. Bezerra de Mello, Class. Quantum Grav. **11**, 1415 (1994).

[15] G. Cognola, K. Kirsten, and L. Vanzo, Phys. Rev. D **49**, 1029 (1994).

[16] M.E.X. Guimarães and B. Linet, Commun. Math. Phys. **165**, 297 (1994); B. Linet, J. Math. Phys. **36**, 3694 (1995).

[17] E.S. Moreira Jnr, Nucl. Phys. B **451**, 365 (1995).

[18] M. Bordag, K. Kirsten, and S. Dowker, Commun. Math. Phys. **182**, 371 (1996).

[19] D. Iellici, Class. Quantum Grav. **14**, 3287 (1997).

[20] N.R. Khusnutdinov and M. Bordag, Phys. Rev. D **59**, 064017 (1999).

[21] J. Spinelly and E.R. Bezerra de Mello, Class. Quantum Grav. **20**, 873 (2003); J. Spinelly and E.R. Bezerra de Mello, J. High Energy Phys. **09**, 005 (2008).

[22] V.B. Bezerra and N.R. Khusnutdinov, Class. Quantum Grav. **23**, 3449 (2006).

[23] Yu.A. Sitenko and N.D. Vlasii, Class. Quantum Grav. **26**, 195009 (2009).

[24] E.R. Bezerra de Mello, Class. Quantum Grav. **27**, 095017 (2010).

[25] E.R. Bezerra de Mello and A.A. Saharian, Phys. Lett. B **642**, 129 (2006); E.R. Bezerra de Mello and A.A. Saharian, Phys. Rev. D **78**, 045021 (2008).

[26] E.R. Bezerra de Mello and A.A. Saharian, J. High Energy Phys. **04**, 046 (2009); E.R. Bezerra de Mello and A.A. Saharian, J. High Energy Phys., in press, arXiv:1006.0224.

[27] V.M. Mostepanenko, N.N. Trunov, *The Casimir Effect and Its Applications* (Clarendon, Oxford, 1997); K.A. Milton, *The Casimir Effect: Physical Manifestation of Zero-Point Energy* (World Scientific, Singapore, 2002); V.A. Parsegian, *Van der Waals Forces* (Cambridge University Press, Cambridge, 2005); M. Bordag, G.L. Klimchitskaya, U. Mohideen, and V.M. Mostepanenko, *Advances in the Casimir Effect* (Oxford University Press, Oxford, 2009).

[28] G.L. Klimchitskaya, U. Mohidden, and V.M. Mostepanenko, Rev. Mod. Phys. **81**, 1827 (2009).

[29] I. Brevik and T. Toverud, Class. Quantum Grav. **12**, 1229 (1995).

[30] E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, and A.S. Tarloyan, Phys. Rev. D **74**, 025017 (2006).

[31] E.R. Bezerra de Mello, V.B. Bezerra, and A.A. Saharian, Phys. Lett. B **645**, 245 (2007).

[32] E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, and A.S. Tarloyan, Phys. Rev. D **78**, 105007 (2008).

[33] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. **140**, 372 (1982); A.J. Niemi and G.W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983); R. Jackiw, Phys. Rev. D **29**, 2375 (1984); A.N. Redlich, Phys. Rev. D **29**, 2366 (1984); M.B. Paranjape, Phys. Rev. Lett. **55**, 2390 (1985); D. Boyanovsky and R. Blankenbecler, Phys. Rev. D **31**, 3234 (1985); R. Blankenbecler and D. Boyanovsky, Phys. Rev. D **34**, 612 (1986).

[34] T. Jaroszewicz, Phys. Rev. D **34**, 3128 (1986).

[35] E.G. Flekkøy and J.M. Leinaas, Int. J. Mod. Phys. A **6**, 5327 (1991).

[36] H. Li, D.A. Coker, and A.S. Goldhaber, Phys. Rev. D **47**, 694 (1993).

[37] V.P. Gusynin, V.A. Miransky and L.A. Shovkovy, Phys. Rev. D **52**, 4718 (1995); R.R. Parwani, Phys. Lett. B **358**, 101 (1995).

[38] Yu.A. Sitenko, Phys. At. Nucl. **60**, 2102 (1997); Yu.A. Sitenko, Phys. Rev. D **60**, 125017 (1999).

[39] G.V. Dunne, Topological Aspects of Low Dimensional Systems (Springer, Berlin, 1999).

[40] A.H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, and A.K. Geim, Rev. Mod. Phys. **81**, 109 (2009).

[41] S. Bellucci and A.A. Saharian, Phys. Rev. D **79**, 085019 (2009); S. Bellucci and A.A. Saharian, Phys. Rev. D **80**, 105003 (2009); S. Bellucci, A.A. Saharian, and V.M. Bardeghyan, arXiv:1002.1391.

[42] Yu.A. Sitenko and N.D. Vlasii, Low Temp. Phys. **34**, 826 (2008).

[43] S. Leseduarte and A. Romeo, Commun. Math. Phys. **193**, 317 (1998).

[44] C.G. Beneventano, M. De Francia, K. Kirsten, and E.M. Santangelo, Phys. Rev. D **61**, 085019 (2000); M. De Francia and K. Kirsten, Phys. Rev. D **64**, 065021 (2001).

[45] P. de Sousa Gerbert and R. Jackiw, Commun. Math. Phys. **124**, 229 (1989); P. de Sousa Gerbert, Phys. Rev. D **40**, 1346 (1989); Yu.A. Sitenko, Ann. Phys. **282**, 167 (2000).

[46] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and Series* (Gordon and Breach, New York, 1986), Vol. 2.

[47] M.F. Lin and D.S. Chuu, Phys. Rev. B **57**, 6731 (1998); S. Latil, S. Roche, and A. Rubio, Phys. Rev. B **67**, 165420 (2003).

[48] A.A. Saharian and E.R. Bezerra de Mello, J. Phys. A: Math. Gen. **37**, 3543 (2004).

[49] A.A. Saharian, *The Generalized Abel-Plana Formula with Applications to Bessel Functions and Casimir Effect* (Yerevan State University Publishing House, Yerevan, 2008); Preprint ICTP/2007/082; arXiv:0708.1187.

[50] P.E. Lammert and V.H. Crespi, Phys. Rev. Lett. **85**, 5190 (2000); A. Cortijo and M.A.H. Vozmediano, Nucl. Phys. B **763**, 293 (2007); Yu.A. Sitenko and N.D. Vlasii, Nucl. Phys. B **787**, 241 (2007); C. Furtado, F. Moraes, and A.M.M. Carvalho, Phys. Lett. A **372**, 5368 (2008); A. Jorio, G. Dresselhaus and M.S. Dresselhaus, *Carbon Nanotubes: Advanced Topics in the Synthesis, Structure, Properties and Applications* (Springer, Berlin, 2008).

[51] A. Krishnan, et al, Nature **388**, 451 (1997); S.N. Naess, A. Elgsaeter, G. Helgesen and K.D. Knudsen, Sci. Technol. Adv. Mater. **10**, 065002 (2009).

[52] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I.A. Stegun, (Dover, New York, 1972).

[53] G.N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944).

[54] A. Romeo and A.A. Saharian, J. Phys. A **35**, 1297 (2002).